

Precoding-Based Network Alignment For Three Unicast Sessions

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Abstract—We consider the problem of network coding across three unicast sessions over a directed acyclic graph. We consider a SISO scenario, in the sense that each source and receiver is connected to the network through a single edge. We adapt a precoding-based interference alignment technique, originally developed for the wireless interference channel, to the network setting. We refer to this approach as precoding-based network alignment (PBNA). Similarly to the wireless setting, PBNA asymptotically achieves half the minimum cut. Different from the wireless setting, network topology may introduce dependencies between elements of the transfer matrix, which we refer to as coupling relations, and can potentially make PBNA infeasible. The goal of this paper is to characterize the classes of networks for which PBNA is feasible, by identifying and interpreting the minimal set of coupling relations introduced by network topology. To this end, first, we identify graph-related properties of transfer functions, which are essential in identifying the minimal set of coupling relations. Then, using two graph-related properties and a degree-counting technique, we greatly reduce the set of possible coupling relations to just three. Finally, we interpret the three coupling relations in terms of network topology and present a polynomial-time algorithm to check the feasibility of PBNA.

Index Terms—network coding, multiple unicasts, interference alignment.

I. INTRODUCTION

UNICAST flows are the dominant form of traffic in most wired and wireless networks today. The problem of designing and adopting communication protocols, that utilize network resources in an efficient way and result in throughput close to network capacity, has been an important topic of research and speculation. Ever since the development of *network coding* and its success in characterizing the achievable throughput for single multicast scenario [1] [2], there has been hope that the framework can be extended to give useful answers to network capacity problems (namely inter-session network coding), especially the practical setting of networks with *multiple unicasts*. Indeed, there have been limited successes in this domain, such as the derivation of a sufficient condition for linear network coding to achieve the maximal throughput, in networks with multiple unicast sessions [3] [4]. However, scalar or even vector linear network coding [5] [6] [7] alone has been shown to be inadequate in characterizing

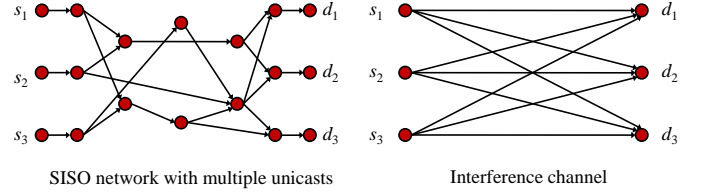


Fig. 1. Analogy between a SISO network employing linear network coding and a wireless interference channel, each with three unicast sessions (s_i, d_i) , $i = 1, 2, 3$. Both these systems can be treated as linear transform systems and are amenable to interference alignment techniques.

the limits of inter-session network coding [8], which includes the network setup with multiple unicasts. Losing the linear coding formulation leaves the problem somewhat unstructured, and that has stunted the progress in obtaining improved rates or even guarantees for a broad class of networks.

In this paper, we consider the problem of linear network coding for networks with multiple unicast sessions, represented by a directed acyclic graph (DAG). Similarly to the SISO wireless channel, we consider a class of SISO networks, where each source and receiver is connected to the network through a single edge (see Fig. 1a). The use of linear network coding provides a linear transfer function representation for a network with respect to its unicast streams [3] [9]. As discussed in [3], the “interference” caused by one unicast stream can significantly affect the achievable rates of other streams. [3] proceeds to develop a sufficient but highly restrictive condition for “interference-free” transmission to be possible in a network with multiple unicast streams. Due to the restrictive condition, it is generally very difficult to find network codes that result in an interference-free framework. In fact, as proved in [10], the problem of finding linear network codes with given rate requirements for multiple unicasts is NP-hard. Thus, only sub-optimal and heuristic methods are known today, e.g., methods based on linear optimization [11] [12] and evolutionary approaches [13].

As opposed to the interference-free framework [3], we present another alternative approach, which allows the existence of interferences at each receiver. Our idea originates from the observation that under the linear network coding framework, a SISO network with multiple unicast sessions essentially mimic the wireless interference channel (Fig. 1). This analogy between networks and wireless interference channel enables us to apply the technique of *precoding-based interfer-*

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ence alignment, originally introduced by Cadambe and Jafar [14] for wireless interference channels, to wired networks. We refer to this approach as *precoding-based network alignment*, or PBNA for short. The use of PBNA significantly simplifies the code design problem of network coding, as all the encoding and decoding operations are predetermined *regardless* of the network topology. Moreover, using this approach, each unicast session is capable of achieving a transmission rate close to one half of the minimum cut between each sender and its corresponding receiver, which is better than what is achieved through time-sharing (i.e., the unicast sessions operate one at a time in a cyclic fashion). Here, we define the minimum cut in the information-theoretic sense [15] – this serves as the upper bound on the achievable rate (network capacity) for any unicast stream.

As mentioned above, the situation of several unicast sessions over a wireline network can be thought of as being equivalent to supporting the same unicast sessions over a wireless interference channel with the same linear transfer functions. Network coding across sessions at intermediate nodes emulates superposition in the wireless channel. Essentially, the entire network can be viewed as a channel, albeit a channel that is not given by nature, as it is the case in wireless, but determined by our routing and coding decisions. This has the advantage that it allows us to control the channel. However, it also has the disadvantage that it introduces spatial-correlation between end-to-end paths that share links among themselves. This correlation is not present in wireless channels with high probability. This is a very important difference between the setup of multiple unicasts vs. traditional wireless interference channel; in our problem, there may be dependencies between elements of the transfer matrix (also called *transfer functions*) introduced by the graph structure, called *coupling relations*, which might make PBNA infeasible in some networks [16]. In contrast, the channel gains in wireless channel, being inherently independent from each other, make precoding-based interference alignment feasible [14] with high probability. Another difference is the fact that while channel gains in wireless interference channel come from the real or complex field, network coding for multiple unicasts has an algebraic flavor to it since the network transfer functions and packet operations are in some finite field domain. Therefore, traditional interference alignment techniques for interference channels cannot be directly applied. In this work, we develop systematic alignment approaches and algorithms for network coding across multiple unicasts, and we characterize their feasibility depending on the network structure. Note that the interplay of multiple sessions in a general network setup can be very complicated with arbitrary correlation between the session streams. As such, we focus on a useful special case: SISO networks with three unicast sessions; this is the smallest, yet non-trivial, instance of the problem and can be used as a building block and better understanding for network coding across multiple unicasts.

Main Contributions. The goal of this paper is to fully characterize the networks, for which PBNA is feasible, by identifying the minimal set of coupling relations that make PBNA infeasible, and interpreting these coupling relations in

terms of network topology. We make the following contributions:

- 1) We show that network topology introduces special properties to the transfer function, which we refer to as graph-related properties.
- 2) Using two graph-related properties and a simple degree-counting technique, in conjunction with the results of [17], we identify the minimal set of coupling relations existing in graphs, the presence of which makes PBNA infeasible.
- 3) We interpret the coupling relations in terms of network topology. Based on these interpretations, we present a polynomial-time algorithm to check the feasibility of PBNA.

The rest of this paper is organized as follows. In Section II, we review related work. In Section III, we show how to apply precoding-based interference alignment to the network setting. In Section IV, we present an overview of our main results regarding the feasibility conditions of PBNA. In Section V, we present two graph-related properties of transfer functions, which play important roles in identifying the minimal set of coupling relations. In Section VI, we discuss the feasibility conditions of PBNA. In Section VII, we provide interpretations of the coupling relations and a polynomial-time algorithm to check the feasibility of PBNA. Section VIII concludes the paper and summarizes future work.

II. RELATED WORK

A. Network Coding

Network coding was first proposed to achieve optimal throughput for single multicast scenario [1] [2] [3], which is a special case of intra-session network coding. The rate region for this setting can be easily calculated by using linear programming formulations [18]. Moreover, the code design for this scenario is fairly simple: Either a polynomial-time algorithm [19] can be used to achieve the optimal throughput in a deterministic manner, or a random network coding scheme [20] can be used to achieve the optimal throughput with high probability.

In contrast, for inter-session network coding, which includes the practical case of multiple unicasts, there have been only limited progresses. It was observed that there exist networks in which network coding significantly outperforms routing schemes in terms of transmission rate [4]. However, there exist only approximation methods to characterize the rate region for this setting [21]. Moreover, it was proven that finding linear network codes for this setting is NP-hard [10]. Therefore, only sub-optimal and heuristic methods exist to construct linear network code for this setting. For example, Ratnakar et al. [11] considered coding pairs of flows using poison-antidote butterfly structures and packing a network using these butterflies to improve throughput; Draskov et al. [12] further proposed a linear programming-based method to find butterfly substructures in the network; Ho et al. [22] developed online and offline back pressure algorithms for finding approximately throughput-optimal network codes within the class of network codes restricted to XOR coding between pairs of flows;

Effros et al. [23] described a tiling approach for designing network codes for wireless networks with multiple unicast sessions on a triangular lattice; Kim et al. [13] proposed an evolutionary approach to construct linear code. Unfortunately, most of these approaches don't provide any guarantee in terms of performance. Moreover, most of these approaches are concerned about finding network codes by jointly considering code assignment and network topology at the same time. In contrast, our approach is oblivious to network topology in the sense that the design of encoding/decoding schemes is isolated from network topology, and is predetermined regardless of network topology. The isolation of code design from network topology greatly simplifies the code design of PBNA.

B. Interference Alignment

The original concept of precoding-based interference alignment was first proposed by Cadambe and Jafar [14] to achieve the optimal degree of freedom (DoF) for K-user wireless interference channel. After that, various approaches to interference alignment have been proposed. For example, Nazer et al. proposed ergodic interference alignment [24]; Bresler, Parekh and Tse proposed lattice alignment [25]; Jafar introduced blind alignment [26] for the scenarios where the actual channel coefficient values are entirely unknown to the transmitters; Maddah-Ali and Tse proposed retrospective interference alignment [27] which exploits only delayed CSIT. Interference alignment has been applied to a wide variety of scenarios, including K-user wireless interference channel [14], compound broad channel [28], cellular networks [29], relay networks [30], and wireless networks supported by a wired backbone [31]. Recently, it was shown that interference alignment can be used to achieve exact repair in distributed storage systems [32] [33].

C. Network Alignment

The idea of applying interference alignment to network coding for multiple unicasts was first introduced by Das et al. [34] [16], where the authors proposed an algebraic formulation of the feasibility conditions for PBNA for three unicast sessions. Ramakrishnan et al. observed that the feasibility of PBNA depends on network topology, and conjectured that the feasibility conditions proposed in [34] can be simplified to just two rational functions [35]. Han et al. [17] further confirmed this conjecture for the simple case of $n = 1$.

The main difference between this paper and prior work is that we fully characterize the class of SISO networks for which PBNA is feasible, by identifying the minimal set of coupling relations, the presence of which invalidates the use of PBNA, and interpreting those relations in terms of network topology. One should note that there are potentially many multiple unicast network setups, where throughputs greater than half the mincut per unicast session may be achievable through linear or non-linear network coding schemes. Therefore, in general, our scheme is not optimal in the sense of achieving the maximal throughput. But this "caveat" is not specific to our approach alone.

Our work is closely related to identifying algebraic properties of transfer functions which are closely related to network topology. Some of recent work also focused on this topic: Ebrahimi and Fragouli [36] found that the structure of a network polynomial, which is the product of the determinants of all transfer matrices, can be described in terms of certain subgraph structures; Weifei et al. [37] proposed the Edge-Reduction Lemma which makes connections between cut sets and the row and column spans of the transfer matrices.

III. APPLYING PRECODING-BASED INTERFERENCE ALIGNMENT TO NETWORKS

The basic idea of network alignment is that under linear network coding, the network behaves like a linear system, similar to the wireless interference channel.' [34] This analogy enables us to borrow some techniques, such as interference alignment [14], which were originally developed for the wireless interference channel, to the network setting. In this section, we first illustrate this analogy by revisiting some basic concepts of linear network coding. Next, we show how to apply precoding-based interference alignment in the context of networks by revisiting the alignment scheme proposed by Cadambe and Jafar [14]. Then, we formulate the feasibility conditions of PBNA, in terms of transfer functions; this forms the basis for our further discussion. Finally, we discuss some important questions regarding the feasibility conditions of PBNA.

A. Treating Networks as Wireless Interference Channel

In order to provide a concise and clear description of our ideas, we consider the simplest non-trivial SISO network scenario of three unicast sessions, where each sender and receiver is connected to the network via an edge. We assume the network is represented by a delay-free directed acyclic graph $G = (V, E)$, where V is the set of nodes and E the set of edges. Without loss of generality, each edge has unit capacity, i.e., can transmit one symbol of finite field \mathbb{F}_q in a unit time, and represents an error-free channel. For the i th unicast session ($i = 1, 2, 3$), let s_i and d_i denote its sender and receiver respectively. For the i th unicast session, let σ_i and τ_i denote the edges which link the s_i and d_i to the network respectively. σ_i and τ_i are referred to as the sender edge and receiver edge for the i th unicast session. For $e \in E$, let $head(e)$ and $tail(e)$ denote the head and tail of e respectively.

Under linear network coding [2] [3], the symbol transmitted along edge e , denoted by Y_e , is a linear combination of incoming symbols at $tail(e)$,

$$Y_e = \begin{cases} X_i & \text{If } e = \sigma_i; \\ \sum_{head(e')=tail(e)} x_{e'e} Y_{e'} & \text{Otherwise.} \end{cases}$$

where X_i represents the source symbol transmitted at s_i ; $x_{e'e}$ is a variable, which takes values from \mathbb{F}_q and represents the coding coefficient used to combine the incoming symbol along e' into the symbol along e . Due to the linear operations at each edge, the output at τ_i is a mixture of source symbols [3],

$$Z_i = m_{1i}(\mathbf{x})X_1 + m_{2i}(\mathbf{x})X_2 + m_{3i}(\mathbf{x})X_3 \quad (1)$$

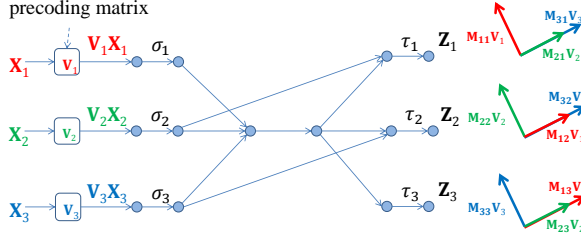


Fig. 2. Applying precoding-based interference alignment to the network setting: At each sender edge σ_i ($i = 1, 2, 3$), the input vector \mathbf{X}_i is first encoded into $2n + 1$ symbols through the precoding matrix \mathbf{V}_i ; then the encoded symbols are transmitted through the network in $2n + 1$ time slots; at each receiver edge τ_i , the undesired symbols are aligned into a single linear space, which is linearly independent from the linear space spanned by the desired signals, such that the receiver can decode all the desired symbols.

where the linear coefficient $m_{ji}(\mathbf{x})$ ($j = 1, 2, 3$) is called the *transfer function* from σ_j to τ_i . According to [3], $m_{ji}(\mathbf{x})$ is a multivariate polynomial in terms of all the coding variables $\mathbf{x} = (x_{ee'} : e, e' \in E', \text{head}(e) = \text{tail}(e'))$, with each monomial corresponding to a path from σ_j to τ_i :

$$m_{ji}(\mathbf{x}) = \sum_{P \in \mathcal{P}_{ji}} t_P(\mathbf{x}) \quad (2)$$

where \mathcal{P}_{ji} denotes the set of paths from σ_j to τ_i , and $t_P(\mathbf{x})$ is the product of all the coding variables along path P and represents a distinct monomial in $m_{ji}(\mathbf{x})$.

Note that Eq. (1) is analogous to that of a wireless interference channel¹, which is shown below:

$$U_i = H_{1i}W_1 + H_{2i}W_2 + H_{3i}W_3 + N_i \quad i = 1, 2, 3 \quad (3)$$

where W_j , H_{ji} , U_i , and N_i ($j = 1, 2, 3$) are all complex numbers, representing the transmitted signal at sender j , the channel gain from sender j to receiver i , the received signal at receiver j , and the noise term respectively. Apparently, X_j 's ($j \neq i$) play the roles of interfering signals, and transfer functions play the roles of channel gains. Hence, we can treat the network as a mimic “wireless interference channel,” and utilize interference alignment in the network. Yet, despite this formal similarity, there are two major differences between the two settings. First, in networks, the transmitted signals belong to a finite field \mathbb{F}_q , whereas in wireless channel, the transmitted signals usually belong to the field of real numbers \mathbb{R} or complex numbers \mathbb{C} , both of which contain an infinite number of elements. Moreover, in networks, the coefficients of received signals are introduced by the linear operations in the middle of the network, and are polynomials in terms of the coding variables in the network, whereas in wireless channel, the channel gains are introduced by nature, and are inherently structureless. Indeed, as we will see later, these two differences greatly affect the feasibility of PBNA.

B. Precoding-Based Network Alignment

Now we explain how to apply precoding-based interference alignment to the above network setting by revisiting the align-

ment scheme proposed by Cadambe and Jafar [14]. We assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros. The case where some $m_{ij}(\mathbf{x})$'s ($i \neq j$) are zeros can be dealt with similarly, and is deferred to Section VI. We use Fig. 2 to illustrate the basic idea of PBNA. As shown in this figure, at each sender edge σ_i ($i = 1, 2, 3$), we first group the source symbols to be transmitted into a vector $\mathbf{X}_i = (X_i^1, \dots, X_i^{k_i(n)})^T$, where n is a positive integer, and $k_i(n)$ equals $n + 1$ for $i = 1$, and n for $i = 2, 3$. We then use *precoding matrix* \mathbf{V}_i to encode \mathbf{X}_i into $2n + 1$ symbols, which are then transmitted through the network in $2n + 1$ time slots. It is easy to see that \mathbf{V}_1 is a $(2n + 1) \times (n + 1)$ matrix, and $\mathbf{V}_2, \mathbf{V}_3$ are both $(2n + 1) \times n$ matrices. As a result, at each receiver edge τ_i , the output vector, denoted by \mathbf{Z}_i , is a combination of input vectors:

$$\mathbf{Z}_i = \mathbf{M}_{1i}\mathbf{V}_1\mathbf{X}_1 + \mathbf{M}_{2i}\mathbf{V}_2\mathbf{X}_2 + \mathbf{M}_{3i}\mathbf{V}_3\mathbf{X}_3$$

where \mathbf{M}_{ji} ($j = 1, 2, 3$) is a diagonal matrix, with the (k, k) element being the transfer function $m_{ji}(\mathbf{x}^k)$, where \mathbf{x}^k represents the vector of the coding variables for the k th time slot. In general, \mathbf{V}_i may still contain variables. Let ξ denote the vector consisting of all the variables in $\mathbf{x}^1, \dots, \mathbf{x}^{2n+1}$ and $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$. We require the following conditions are satisfied under some assignment of values to ξ [14]:

$$\begin{aligned} \mathcal{A}_1 &: \text{span}(\mathbf{M}_{21}\mathbf{V}_2) = \text{span}(\mathbf{M}_{31}\mathbf{V}_3) \\ \mathcal{A}_2 &: \text{span}(\mathbf{M}_{32}\mathbf{V}_3) \subseteq \text{span}(\mathbf{M}_{12}\mathbf{V}_1) \\ \mathcal{A}_3 &: \text{span}(\mathbf{M}_{23}\mathbf{V}_2) \subseteq \text{span}(\mathbf{M}_{13}\mathbf{V}_1) \\ \mathcal{B}_1 &: \text{rank}(\mathbf{M}_{11}\mathbf{V}_1 \quad \mathbf{M}_{21}\mathbf{V}_2) = 2n + 1 \\ \mathcal{B}_2 &: \text{rank}(\mathbf{M}_{12}\mathbf{V}_1 \quad \mathbf{M}_{22}\mathbf{V}_2) = 2n + 1 \\ \mathcal{B}_3 &: \text{rank}(\mathbf{M}_{13}\mathbf{V}_1 \quad \mathbf{M}_{33}\mathbf{V}_3) = 2n + 1 \end{aligned}$$

In the above conditions, for a matrix \mathbf{E} , we use $\text{span}(\mathbf{E})$ to denote the linear space spanned by the column vectors contained in \mathbf{E} . $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are the *alignment conditions* and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are the *rank conditions*. Basically, each alignment condition guarantees that the undesired symbols or interferences at each receiver are mapped into a single linear space, such that the dimension of received symbols or the number of unknowns is decreased. Moreover, each rank condition guarantees that the linear space spanned by the interferences is linearly independent from that spanned by the desired symbols, and thus each receiver can decode the desired symbols from the received symbols. In summary, when all these conditions are satisfied, the three unicast sessions can achieve a rate tuple $\mathbf{R}_n^* \triangleq (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$, which approaches $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ as $n \rightarrow \infty$. In this case, we say that \mathbf{R}_n^* is feasible through PBNA.

Note that the alignment conditions and rank conditions are all defined on the field of rational functions $\mathbb{F}_q(\xi)$. This poses no problem for the alignment conditions, because whenever they are satisfied on $\mathbb{F}_q(\xi)$, they are automatically satisfied under *any* assignment of values to ξ . However, this is not always true for the rank conditions, unless the size of \mathbb{F}_q is sufficiently large. In contrast, in wireless channel, the alignment and rank conditions are all defined on $\mathbb{R}(\xi')$ or $\mathbb{C}(\xi')$, where ξ' consists of the channel gains for the $2n + 1$

¹The wireless interference channel that we consider here has only one sub-channel.

time slots. Due to the infiniteness of \mathbb{R} and \mathbb{C} , as long as the rank conditions are satisfied on $\mathbb{R}(\xi')$ or $\mathbb{C}(\xi')$, they are satisfied *almost surely* under a random assignment of values to ξ' in the context of wireless channel [14]. This indicates that the size of \mathbb{F}_q does affect the feasibility of PBNA. To avoid this issue, we assume that the size of \mathbb{F}_q is sufficiently large. Let $\phi_i(\xi) \in \mathbb{F}_q(\xi)$ denote the determinant of the matrix involved in \mathcal{B}_i . Due to the Schwartz-Zippel Lemma [38], if the rank conditions are satisfied on $\mathbb{F}_q(\xi)$, i.e., $\phi(\xi) = \phi_1(\xi)\phi_2(\xi)\phi_3(\xi)$ is a non-zero polynomial, we can find an assignment of values to ξ with high probability such that $\phi(\xi)$ is non-zero under this assignment.

C. Algebraic Formulation of the Feasibility Conditions

As shown previously, the feasibility conditions of PBNA, namely the alignment conditions and the rank conditions, are all expressed in matrix forms, which are difficult to analyze and reveal little about their connection with the network topology. For these reasons, before proceeding, we reformulate the feasibility conditions in terms of transfer functions. To this end, we first reformulate the alignment conditions as follows:

$$\begin{aligned}\mathcal{A}'_1 : \mathbf{M}_{21}\mathbf{V}_2 &= \mathbf{M}_{31}\mathbf{V}_3\mathbf{A} \\ \mathcal{A}'_2 : \mathbf{M}_{32}\mathbf{V}_3 &= \mathbf{M}_{12}\mathbf{V}_1\mathbf{B} \\ \mathcal{A}'_3 : \mathbf{M}_{23}\mathbf{V}_2 &= \mathbf{M}_{13}\mathbf{V}_1\mathbf{C}\end{aligned}$$

where \mathbf{A} is an $n \times n$ invertible matrix, and \mathbf{B} , \mathbf{C} are both $(n+1) \times n$ matrices with rank n . A direct consequence of \mathcal{A}'_2 and \mathcal{A}'_3 is that the precoding matrices are not independent from each other: Both \mathbf{V}_2 and \mathbf{V}_3 are determined by \mathbf{V}_1 through the following equations:

$$\mathbf{V}_2 = \mathbf{M}_{13}\mathbf{M}_{23}^{-1}\mathbf{V}_1\mathbf{C} \quad \mathbf{V}_3 = \mathbf{M}_{12}\mathbf{M}_{32}^{-1}\mathbf{V}_1\mathbf{B} \quad (4)$$

Substituting the above equations into \mathcal{A}'_1 , the three alignment conditions can be further condensed into a single equation:

$$\mathbf{T}\mathbf{V}_1\mathbf{C} = \mathbf{V}_1\mathbf{B}\mathbf{A} \quad (5)$$

where $\mathbf{T} = \mathbf{M}_{13}\mathbf{M}_{21}\mathbf{M}_{32}\mathbf{M}_{12}^{-1}\mathbf{M}_{23}^{-1}\mathbf{M}_{31}^{-1}$. Eq. (5) suggests that alignment conditions introduce self-constraint on \mathbf{V}_1 . Thus, in general, we cannot choose \mathbf{V}_1 freely. Indeed, Eq. (5) is also the major restriction on \mathbf{V}_1 . Finally, using Eq. (4) and Eq. (5), the rank conditions are transformed into the following equivalent equations:

$$\begin{aligned}\mathcal{B}'_1 : \text{rank}(\mathbf{V}_1 \quad \mathbf{P}_1\mathbf{V}_1\mathbf{C}) &= 2n+1 \\ \mathcal{B}'_2 : \text{rank}(\mathbf{V}_1 \quad \mathbf{P}_2\mathbf{V}_1\mathbf{C}) &= 2n+1 \\ \mathcal{B}'_3 : \text{rank}(\mathbf{V}_1 \quad \mathbf{P}_3\mathbf{V}_1\mathbf{C}\mathbf{A}^{-1}) &= 2n+1\end{aligned}$$

where $\mathbf{P}_1 = \mathbf{M}_{13}\mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{23}^{-1}$, $\mathbf{P}_2 = \mathbf{M}_{13}\mathbf{M}_{22}\mathbf{M}_{12}^{-1}\mathbf{M}_{23}^{-1}$, and $\mathbf{P}_3 = \mathbf{M}_{21}\mathbf{M}_{33}\mathbf{M}_{23}^{-1}\mathbf{M}_{31}^{-1}$. Thus, we have simplified the feasibility conditions into four equations, i.e., Eq. (5) and $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$. Note that all these equations are formulated in terms of \mathbf{V}_1 and \mathbf{P}_i . We summarize this result into the following lemma:

Lemma III.1. Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros. \mathbf{R}_n^* is feasible through PBNA if and only if 1) Eq. (5) is satisfied, and 2) $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ are satisfied.

We further define the following functions:

$$\begin{aligned}p_1(\mathbf{x}) &= \frac{m_{13}(\mathbf{x})m_{21}(\mathbf{x})}{m_{11}(\mathbf{x})m_{23}(\mathbf{x})} & p_2(\mathbf{x}) &= \frac{m_{13}(\mathbf{x})m_{22}(\mathbf{x})}{m_{12}(\mathbf{x})m_{23}(\mathbf{x})} \\ p_3(\mathbf{x}) &= \frac{m_{21}(\mathbf{x})m_{33}(\mathbf{x})}{m_{23}(\mathbf{x})m_{31}(\mathbf{x})} & \eta(\mathbf{x}) &= \frac{m_{13}(\mathbf{x})m_{21}(\mathbf{x})m_{32}(\mathbf{x})}{m_{12}(\mathbf{x})m_{23}(\mathbf{x})m_{31}(\mathbf{x})}\end{aligned} \quad (6)$$

Clearly, $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$ form the elements along the diagonals of \mathbf{P}_i and \mathbf{T} respectively.

Next, we reformulate the feasibility conditions in terms of $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$. To this end, we need to know the internal structure of \mathbf{V}_1 . We distinguish the following two cases:

Case I: $\eta(\mathbf{x})$ is non-constant, and thus \mathbf{T} is not an identity matrix. For this case, Eq. (5) becomes non-trivial, and we cannot choose \mathbf{V}_1 freely. We use the following precoding matrices, which were first proposed by Cadambe and Jafar [14]:

$$\mathbf{V}_1^* = (\mathbf{w} \quad \mathbf{T}\mathbf{w} \quad \cdots \quad \mathbf{T}^n\mathbf{w}) \quad (7)$$

$$\mathbf{V}_2^* = \mathbf{M}_{13}\mathbf{M}_{23}^{-1}(\mathbf{w} \quad \mathbf{T}\mathbf{w} \quad \cdots \quad \mathbf{T}^{n-1}\mathbf{w}) \quad (8)$$

$$\mathbf{V}_3^* = \mathbf{M}_{12}\mathbf{M}_{32}^{-1}(\mathbf{T}\mathbf{w} \quad \mathbf{T}^2\mathbf{w} \quad \cdots \quad \mathbf{T}^n\mathbf{w}) \quad (9)$$

where \mathbf{w} is a column vector of $2n+1$ ones. It is straightforward to verify that the above precoding matrices satisfy the alignment conditions. Note that the above precoding matrices correspond to the configuration where $\mathbf{A} = \mathbf{I}_n$, \mathbf{C} consists of the left n columns of \mathbf{I}_{n+1} , and \mathbf{B} the right n columns of \mathbf{I}_{n+1} .

In order to reformulate the rank conditions, we consider the following matrix,

$$\mathbf{H} = \begin{pmatrix} f_1(\mathbf{y}^1) & f_2(\mathbf{y}^1) & \cdots & f_r(\mathbf{y}^1) \\ f_1(\mathbf{y}^2) & f_2(\mathbf{y}^2) & \cdots & f_r(\mathbf{y}^2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(\mathbf{y}^r) & f_2(\mathbf{y}^r) & \cdots & f_r(\mathbf{y}^r) \end{pmatrix}$$

where $f_i(\mathbf{y})$ ($i = 1, 2, \dots, r$) is a rational function in $\mathbb{F}_q(\mathbf{y})$, and the j th row of \mathbf{H} is simply a repetition of the vector $(f_1(\mathbf{y}), \dots, f_r(\mathbf{y}))$, with \mathbf{y} being replaced by \mathbf{y}^j . Due to the particular structure of \mathbf{H} , the problem of checking whether \mathbf{H} is full rank can be simplified to checking whether $f_1(\mathbf{y}), \dots, f_r(\mathbf{y})$ are linearly independent, as stated in the following lemma. Here, $f_1(\mathbf{y}), \dots, f_r(\mathbf{y})$ are said to be linearly independent, if for any scalars $a_1, \dots, a_r \in \mathbb{F}_q$, which are not all zeros, $a_1f_1(\mathbf{y}) + \cdots + a_rf_r(\mathbf{y}) \neq 0$.

Lemma III.2. $\det(\mathbf{H}) \neq 0$ if and only if $f_1(\mathbf{y}), \dots, f_r(\mathbf{y})$ are linearly independent.

Proof: See Theorem 1 of [17]. ■

An important observation is that using the precoding matrices defined in Eq. (7)-(9), all of the matrices involved in $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ have the same form as \mathbf{H} . Specifically, each row of the matrix in \mathcal{B}'_i is of the form:

$$(1 \quad \eta(\mathbf{x}) \quad \cdots \quad \eta^n(\mathbf{x}) \quad p_i(\mathbf{x}) \quad \cdots \quad p_i(\mathbf{x})\eta^{n-1}(\mathbf{x})) \quad (10)$$

Hence, using Lemma III.2, we can quickly derive:

Lemma III.3. Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is non-constant. \mathbf{R}_n^* is feasible through PBNA

if for each $i = 1, 2, 3$,²

$$p_i(\mathbf{x}) \notin \mathcal{S}_n = \left\{ \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} : f(z), g(z) \in \mathbb{F}_q[z], f(z)g(z) \neq 0, \right. \\ \left. \gcd(f(z), g(z)) = 1, d_f \leq n, d_g \leq n-1 \right\} \quad (11)$$

Proof: If Eq. (11) is satisfied, the rational functions in Eq. (10) are linearly independent. Therefore, due to Lemma III.2, condition \mathcal{B}'_i is satisfied. Meanwhile, the precoding matrix \mathbf{V}_1 defined in Eq. (7) satisfies Eq. (5). Hence, \mathbf{R}_n^* is feasible through PBNA by Lemma III.1. ■

Note that each rational function $\frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}_n$ represents a constraint on $p_i(\mathbf{x})$, i.e., $p_i(\mathbf{x}) \neq \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$, the violation of which invalidates the use of the PBNA through the precoding matrices defined in Eq. (7)-(9). Also note that Eq. (11) only guarantees that PBNA is feasible for a fixed value of n , i.e., each unicast session only achieves a transmission rate close to one half of its minimum cut. In order for each unicast session to asymptotically achieve a transmission rate of one half of its minimum cut, we simply combine the conditions of Lemma III.3 for all possible values of n , and get the following result:

Theorem III.1. Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is non-constant. The three unicast sessions can asymptotically achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ through PBNA if for each $i = 1, 2, 3$,

$$p_i(\mathbf{x}) \notin \mathcal{S}' = \left\{ \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} : f(z), g(z) \in \mathbb{F}_q[z], f(z)g(z) \neq 0, \right. \\ \left. \gcd(f(z), g(z)) = 1 \right\} \quad (12)$$

Proof: Clearly, if Eq. (12) is satisfied, \mathbf{R}_n^* is feasible through PBNA for all possible values of n . Thus, each unicast session can asymptotically achieve one half rate as $n \rightarrow \infty$. ■

Case II: $\eta(\mathbf{x})$ is constant, and thus \mathbf{T} is an identity matrix. For this case, Eq. (5) becomes trivial. In fact, we set $\mathbf{B}\mathbf{A} = \mathbf{C}$, and hence Eq. (5) can be satisfied by any arbitrary \mathbf{V}_1 . Specifically, we use the following precoding matrices:

$$\mathbf{V}_1 = (\theta_{ij})_{(2n+1) \times (n+1)} \quad (13)$$

$$\mathbf{V}_2 = \mathbf{M}_{13}\mathbf{M}_{23}^{-1}\mathbf{V}_1\mathbf{I}' \quad (14)$$

$$\mathbf{V}_3 = \mathbf{M}_{12}\mathbf{M}_{32}^{-1}\mathbf{V}_1\mathbf{I}' \quad (15)$$

where θ_{ij} is a variable, taking values from \mathbb{F}_q , and \mathbf{I}' consists of the left n columns of \mathbf{I}_{n+1} . Note that the above precoding matrices correspond to the configuration where $\mathbf{A} = \mathbf{I}_n$, and $\mathbf{B} = \mathbf{C} = \mathbf{I}'$. Clearly, the above precoding matrices satisfy the alignment conditions. Meanwhile, using these precoding matrices, each row of the matrix in \mathcal{B}'_i is of the following form:

$$(\theta_1 \theta_2 \cdots \theta_{n+1} p_i(\mathbf{x})\theta_1 \cdots p_i(\mathbf{x})\theta_n) \quad (16)$$

Hence, using Lemma III.2, we can quickly derive:

²Notation: For two polynomials $f(x)$ and $g(x)$, let $\gcd(f(x), g(x))$ denote their greatest common divisor, and d_f the degree of $f(x)$.

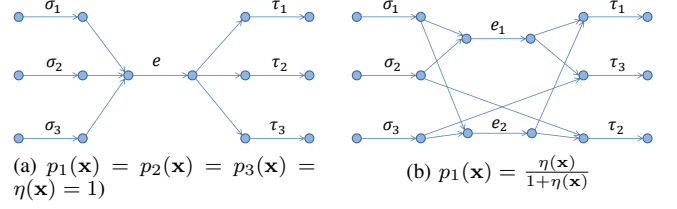


Fig. 3. Examples of realizable coupling relations: The left network realizes the coupling relations $p_i(\mathbf{x}) = \eta(\mathbf{x}) = 1$ such that the conditions of Theorem III.2 are violated; in the right network, $\eta(\mathbf{x}) \neq 1$, but $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$, which violates the conditions of Theorem III.1.

Lemma III.4. Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is constant. \mathbf{R}_n^* is feasible through PBNA if for each $i = 1, 2, 3$, $p_i(\mathbf{x})$ is not constant.

Proof: Apparently, the functions in Eq. (16) are linearly independent, and therefore \mathcal{B}'_i is satisfied due to Lemma III.2. Moreover, Eq. (5) is also satisfied. Thus, \mathbf{R}_n^* is feasible through PBNA by Lemma III.1. ■

Similarly, the above theorem can be easily extended to the asymptotic case:

Theorem III.2. Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is constant. The three unicast sessions can asymptotically achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ through PBNA if for each $i = 1, 2, 3$, $p_i(\mathbf{x})$ is not constant.

Proof: By Theorem III.2, if $p_i(\mathbf{x})$ is non-constant, \mathbf{R}_n^* is feasible through PBNA for any value of n . Thus, each unicast session can asymptotically achieve one half rate as $n \rightarrow \infty$. ■

D. Further Discussion

In the previous section, we reformulated the feasibility conditions in terms of the transfer functions $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$. The critical question is: what is the connection between the reformulated feasibility conditions and network topology?

We start by illustrating that through examples of networks whose structure violates the feasibility conditions. Let's first consider the network shown in Fig. 3a. Due to the bottleneck e , it can be easily verified that $p_1(\mathbf{x}) = p_2(\mathbf{x}) = p_3(\mathbf{x}) = \eta(\mathbf{x}) = 1$, and thus the conditions of Theorem III.2 are violated. Moreover, consider the network shown in Fig. 3b. It can be easily verified that for this network, $\eta(\mathbf{x}) \neq 1$, and $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$. Thus the conditions of Theorem III.1 are violated. Moreover, by exchanging $\sigma_1 \leftrightarrow \sigma_2$ and $\tau_1 \leftrightarrow \tau_2$, we obtain another example, where $p_2(\mathbf{x}) = 1 + \eta(\mathbf{x})$, and thus the conditions of Theorem III.1 are again violated. While the key feature of the first example can be easily identified, it is not obvious what are the defining features of the second example. Nevertheless, both examples demonstrate an important difference between networks and wireless interference channel: In networks, due to the internal structure of transfer functions, network topology might introduce dependence between different transfer functions, e.g., $p_1(\mathbf{x}) = 1$ or $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$; in contrast, in wireless channel, channel gains are inherently structureless, and thus can change *independently*.

Note that the above dependence relations between transfer functions can be generalized to the following form:

$$f(m_{i_1 j_1}(\mathbf{x}), m_{i_2 j_2}(\mathbf{x}), \dots, m_{i_k j_k}(\mathbf{x})) = 0 \quad (17)$$

where $f(z_1, z_2, \dots, z_k)$ is a polynomial in $\mathbb{F}_q[z_1, \dots, z_k]$. We call such relation a *coupling relation*. As shown in Theorem III.1, each rational function $\frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$ represents a coupling relation $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$. Given a coupling relation, if there are networks for which it holds, we say that it is *realizable*.

The existence of coupling relations greatly complicates the feasibility problem of PBNA. As shown previously, most of the coupling relations, such as $p_1(\mathbf{x}) = 1$ and $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$, are harmful to PBNA, because their presence violates the feasibility conditions. The only exception is $\eta(\mathbf{x}) = 1$, which does help simplify the construction of precoding matrices, and thus is beneficial to PBNA. Indeed, as we will see in Section IV, this coupling relation allows interferences to be perfectly aligned at each receiver, and thus each unicast session can achieve one half rate in *exactly* two time slots. Unfortunately, as we will see in Section VII, this coupling relation requires that the network possesses peculiar structure, and is generally unavailable in most networks. For this reason, we will mainly focus on Theorem III.1, which is applicable for most networks. This is in stark contrast with wireless channel, where precoding-based interference alignment is feasible almost surely due to the structureless nature of channel gains [14].

One interesting observation is that *not* all coupling relations are realizable. For example, consider the coupling relation $p_1(\mathbf{x}) = \eta^3(\mathbf{x})$, where both $p_1(\mathbf{x})$ and $\eta(\mathbf{x})$ are non-constants. Let $p_1(\mathbf{x}) = \frac{u(\mathbf{x})}{v(\mathbf{x})}$, $\eta(\mathbf{x}) = \frac{s(\mathbf{x})}{t(\mathbf{x})}$ denote the *unique forms*³ of $p_1(\mathbf{x})$ and $\eta(\mathbf{x})$ respectively. Consider a coding variable $x_{ee'}$ that appears in both $\frac{u(\mathbf{x})}{v(\mathbf{x})}$ and $\frac{s(\mathbf{x})}{t(\mathbf{x})}$. Because the maximum degree of each coding variable in a transfer function is at most one, according to Eq. (6), the maximum of the degrees of $x_{ee'}$ in $u(\mathbf{x})$ and $v(\mathbf{x})$ is at most two. However, it can be easily seen that the maximum of the degrees of $x_{ee'}$ in $s^3(\mathbf{x})$ and $t^3(\mathbf{x})$ is at least three. Therefore, it is impossible that $p_1(\mathbf{x}) = \eta^3(\mathbf{x})$. This example suggests that there exists significant redundancy in the feasibility conditions of Theorem III.1. More formally, it raises the following important question:

Q1: Which coupling relations $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$ are realizable?

The answer to this question allows us to reduce the set \mathcal{S}' defined in Theorem III.1 to its minimal size. For $i = 1, 2, 3$, we define the following set, which represents the minimal set of coupling relations we need to consider:

$$\mathcal{S}'_i = \left\{ \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}' : p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \text{ is realizable} \right\} \quad (18)$$

Then the next important question is:

Q2: Given $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'_i$, what are the defining features of the networks for which this coupling relation holds?

As we will see in the rest of this paper, the answers to Q1 and Q2 both lie in a deeper understanding of the properties of

transfer functions. Intuitively, because each transfer function is defined on a graph, it usually possesses special properties, called *graph-related properties*, which are closely related to the graph structure. The graph-related properties not only allow us to reduce \mathcal{S}' to the minimal set \mathcal{S}'_i , but also enable us to identify the defining features of the networks which realize the coupling relations represented by \mathcal{S}'_i .

In the above discussion, we have overlooked a subtle issue. In the derivation of Theorem III.1, we only consider the precoding matrices defined in Eq. (7)-(9). However, the choices of precoding matrices are not limited to these matrices. In fact, as we will see in Section VI, given different \mathbf{A} , \mathbf{B} , and \mathbf{C} , we can derive different precoding matrix \mathbf{V}_1 such that Eq. (5) is satisfied. This raises the following interesting question:

Q3: Assume some coupling relation $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'_i$ is present in the network. Is it still possible to utilize PBNA via other precoding matrices instead of those defined in Eq. (7)-(9)?

As we will see in Section VI, the answer to this question is negative. The basic idea is that each precoding matrix \mathbf{V}_1 that satisfies Eq. (5) is closely related to the precoding matrix defined in Eq. (7) through a transform equation. Using this transform equation, we can prove that if the precoding matrices cannot be used due to the presence of a coupling relation, then any precoding matrices cannot be used.

IV. OVERVIEW OF MAIN RESULTS

In this section, we state our main results regarding the feasibility of PBNA. Proofs are deferred to Section VI and Appendix B. Since the construction of \mathbf{V}_1 depends on whether $\eta(\mathbf{x})$ is constant, we distinguish two cases.

A. $\eta(\mathbf{x})$ Is Not Constant

Theorem IV.1 (The Main Theorem). *Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is not constant. The three unicast sessions can asymptotically achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ through PBNA if and only if the following conditions are satisfied:*

$$p_1(\mathbf{x}) \notin \mathcal{S}'_1 = \left\{ 1, \eta(\mathbf{x}), \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})} \right\} \quad (19)$$

$$p_2(\mathbf{x}) \notin \mathcal{S}'_2 = \{1, \eta(\mathbf{x}), 1+\eta(\mathbf{x})\} \quad (20)$$

$$p_3(\mathbf{x}) \notin \mathcal{S}'_3 = \{1, \eta(\mathbf{x}), 1+\eta(\mathbf{x})\} \quad (21)$$

Note that in the Main Theorem, we reduce the feasibility conditions of Theorem III.1 to its minimal size, such that each \mathcal{S}'_i ($i = 1, 2, 3$) represents the minimal set of coupling relations that are realizable. Moreover, as we will see in Section VII, each of these coupling relations has a unique interpretation in terms of the network topology. The interpretations of these coupling relations further provide a polynomial-time algorithm to verify the feasibility of PBNA.

Remark: The conditions of the Main Theorem can be understood from the perspective of interference channel. As shown previously, under linear network coding, the network behaves as a 3-user wireless interference channel, where the channel coefficients $m_{ij}(\mathbf{x})$ are all non-zeros. Let \mathbf{H} denote

³For a non-zero rational function $h(\mathbf{y}) \in \mathbb{F}_q(\mathbf{y})$, its unique form is defined as $h(\mathbf{y}) = \frac{f(\mathbf{y})}{g(\mathbf{y})}$, where $f(\mathbf{y}), g(\mathbf{y}) \in \mathbb{F}_q[\mathbf{y}]$ and $\gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$.

the matrix with the (i, j) -element being $m_{ij}(\mathbf{x})$. Suppose that there exists a (k, l) -Minor of \mathbf{H} ($k \neq l$) which equals zero, i.e., $M_{kl}(\mathbf{H}) = 0$. For such a channel, it is known that the sum-rate achieved by the three unicast sessions cannot be more than 1 in the information theoretical sense (see Lemma 1 of [39]), i.e., no precoding-based scheme, linear or non-linear⁴, can achieve more than 1/3 per user. Therefore, given that channel coefficients are always non-zero, the condition that $M_{kl}(\mathbf{H}) \neq 0$ for $k \neq l$ is information theoretically necessary for rate 1/2 feasibility. Note that the conditions $p_1(\mathbf{x}) \neq 1, p_1(\mathbf{x}) \neq \eta(\mathbf{x}), p_1(\mathbf{x}) \neq \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ can be respectively rewritten as follows:

$$\begin{aligned} m_{11}(\mathbf{x}) &\neq \frac{m_{13}(\mathbf{x})m_{21}(\mathbf{x})}{m_{23}(\mathbf{x})} \\ m_{11}(\mathbf{x}) &\neq \frac{m_{12}(\mathbf{x})m_{31}(\mathbf{x})}{m_{32}(\mathbf{x})} \\ m_{11}(\mathbf{x}) &\neq \frac{m_{13}(\mathbf{x})m_{21}(\mathbf{x})}{m_{23}(\mathbf{x})} + \frac{m_{12}(\mathbf{x})m_{31}(\mathbf{x})}{m_{32}(\mathbf{x})} \end{aligned}$$

The other conditions can be reformulated by permuting the indices. Apparently, the first two are equivalent to $M_{32}(\mathbf{H}) \neq 0$ and $M_{23}(\mathbf{H}) \neq 0$ respectively. Hence, they are simply the information theoretic *necessary* conditions, so they must hold for *any* achievable precoding-based scheme. The third condition involves a sum of the first two, which is not necessary information theoretically, but is needed for the precoding-based scheme we utilize. In fact, consider the network shown in Fig. 3b, where the third condition is violated. It is fairly easy to find a linear network code such that 1/2 rate is achieved.

B. $\eta(\mathbf{x})$ Is Constant

In this case, we can choose \mathbf{V}_1 freely by setting $\mathbf{BA} = \mathbf{C}$, and thus the feasibility conditions of PBNA are significantly simplified. Moreover, each unicast session can achieve one half rate in exactly two time slots, as stated in the following theorem:

Theorem IV.2. *Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is constant. The three unicast sessions can achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in exactly two time slots through PBNA if and only if for each $i = 1, 2, 3$, $p_i(\mathbf{x}) \neq 1$.*

V. GRAPH-RELATED PROPERTIES

Since the transfer functions are defined on graphs, they exhibit special graph-related properties introduced by the graph structure. In this section, we present two such properties, namely Linearization Property and Square-Term Property, which play central roles in the proof of the Main Theorem. We defer the proofs of these two properties to Appendix A. The other graph-related properties are related to the interpretation of coupling relations defined in Eq. (19)-(21) and are deferred to Appendix A.

⁴In the precoding-based scheme mentioned here, the encoding/decoding process at each sender/receiver might be linear or non-linear, but the operations at each internal node are linear.

In the following discussion, we consider the general form of the transfer function $p_i(\mathbf{x})$, as defined below

$$h(\mathbf{x}) = \frac{m_{ab}(\mathbf{x})m_{pq}(\mathbf{x})}{m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})} \quad (22)$$

where $a, b, p, q = 1, 2, 3$ and $a \neq p, b \neq q$. Moreover, by the definition of transfer function, the numerator and denominator of $h(\mathbf{x})$ can be expanded respectively as follows:

$$\begin{aligned} m_{ab}(\mathbf{x})m_{pq}(\mathbf{x}) &= \sum_{(P_1, P_2) \in \mathcal{P}_{ab} \times \mathcal{P}_{pq}} t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x}) \\ m_{aq}(\mathbf{x})m_{pb}(\mathbf{x}) &= \sum_{(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}} t_{P_3}(\mathbf{x})t_{P_4}(\mathbf{x}) \end{aligned}$$

Hence, each path pair in $\mathcal{P}_{ab} \times \mathcal{P}_{pq}$ contributes a term in $m_{ab}(\mathbf{x})m_{pq}(\mathbf{x})$, and each path pair in $\mathcal{P}_{aq} \times \mathcal{P}_{pb}$ contributes a term in $m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})$.

A. Linearization Property

The first graph-related property, namely Linearization Property, is stated in the following lemma. According to this property, if $p_i(\mathbf{x}) \neq 1$, it can be transformed into its simplest non-trivial form, i.e., a linear function or the inverse of a linear function, through a partial assignment of values to \mathbf{x} .

Lemma V.1 (Linearization Property). *Assume $h(\mathbf{x})$ is not constant. Let $h(\mathbf{x}) = \frac{u(\mathbf{x})}{v(\mathbf{x})}$ such that $\gcd(u(\mathbf{x}), v(\mathbf{x})) = 1$. Then, we can assign values to \mathbf{x} other than a variable $x_{ee'}$ such that $u(\mathbf{x})$ and $v(\mathbf{x})$ are transformed into either $u(x_{ee'}) = c_1x_{ee'} + c_0$, $v(x_{ee'}) = c_2$ or $u(x_{ee'}) = c_2$, $v(x_{ee'}) = c_1x_{ee'} + c_0$, where c_0, c_1, c_2 are constants in \mathbb{F}_q , and $c_1c_2 \neq 0$.*

The key to the above lemma is to find a subgraph H and consider $h(\mathbf{x})$ restricted to H , i.e., $h(\mathbf{x}_H) = \frac{m_{ab}(\mathbf{x}_H)m_{pq}(\mathbf{x}_H)}{m_{aq}(\mathbf{x}_H)m_{pb}(\mathbf{x}_H)}$, where \mathbf{x}_H consists of the coding variables in H . In fact, due to graph structure, we can always find H such that some variable $x_{ee'}$ appears exclusively in the numerator or the denominator of $h(\mathbf{x}_H)$. Thus, by assigning values to \mathbf{x}_H other than $x_{ee'}$, we can transform $h(\mathbf{x}_H)$ into a linear function or the inverse of a linear function in terms of $x_{ee'}$. Since $h(\mathbf{x}_H)$ can be acquired through a partial assignment to \mathbf{x} , this transformation also holds for the extended graph G' .

B. Square-Term Property

The second property, namely Square-Term Property, is presented in the following lemma. According to this property, the coefficient of $x_{ee'}^2$ in the numerator of $h(\mathbf{x})$ equals its counterpart in the denominator of $h(\mathbf{x})$. Thus, if $x_{ee'}^2$ appears in the numerator of $h(\mathbf{x})$ under some assignment to \mathbf{x} , it must also appear in the denominator of $h(\mathbf{x})$, and vice versa.

Lemma V.2 (Square-Term Property). *Given a coding variable $x_{ee'}$, let $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ be the coefficients of $x_{ee'}^2$ in $m_{ab}(\mathbf{x})m_{pq}(\mathbf{x})$ and $m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})$ respectively. Then $f_1(\mathbf{x}) = f_2(\mathbf{x})$.*

The key idea of the above lemma is that due to graph structure, each path pair which contributes an $x_{ee'}^2$ term in the numerator of $h(\mathbf{x})$ corresponds to another path pair which contributes an equivalent $x_{ee'}^2$ term in the denominator of $h(\mathbf{x})$.

This correspondence relation automatically yields a one-to-one mapping from the $x_{ee'}^2$ terms in the numerator of $h(\mathbf{x})$ to those in the denominator of $h(\mathbf{x})$. Thus, the summation of the $x_{ee'}^2$ terms in the numerator of $h(\mathbf{x})$ equals the summation of the $x_{ee'}^2$ terms in the denominator of $h(\mathbf{x})$, and hence $f_1(\mathbf{x}) = f_2(\mathbf{x})$.

VI. FEASIBILITY CONDITIONS OF PBNA

In this section, we explain the main idea behind the proof of our main results presented in IV. Consistently with Section IV, we distinguish two cases based on whether $\eta(\mathbf{x})$ is constant.

A. $\eta(\mathbf{x})$ Is Not Constant

In this subsection, we first present a simple method to quickly identify a class of networks, for which PBNA is feasible. Then, we explain how to reduce the set \mathcal{S}' defined in Theorem III.1 to the minimal set \mathcal{S}'_i defined in the Main Theorem. This answers the first question we raise in Section III-D, and automatically proves the “if” part of the Main Theorem. Next, we provide an answer to the third question we raise in Section III-D, which actually proves the “only if” part of the Main Theorem.

1) *A Simple Method Based on Theorem III.1:* As shown in Theorem III.1, the set \mathcal{S}' contains an infinite number of rational functions, and thus it is impossible to check the feasibility conditions of Theorem III.1 in practice. Interestingly, the theorem directly yields a simple method to quickly identify a class of networks for which PBNA is feasible. The major idea of the method is to exploit the asymmetry between $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$ in terms of *effective variables*. Here, given a rational function $f(\mathbf{y})$, we define a variable as an effective variable of $f(\mathbf{y})$ if it appears in the unique form of $f(\mathbf{y})$. Let $\mathcal{V}(f(\mathbf{y}))$ denote the set of effective variables of $f(\mathbf{y})$. Intuitively, this asymmetry allows us more freedom to control the values of $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$ such that they can change independently, which makes the network behave more like a wireless channel. The formal description of the method is presented below:

Corollary VI.1. *Assume all $m_{ij}(\mathbf{x})$'s ($i, j = 1, 2, 3$) are non-zeros, and $\eta(\mathbf{x})$ is not constant. Each unicast session can asymptotically achieve one half rate through PBNA if for $i = 1, 2, 3$, $p_i(\mathbf{x}) \neq 1$ and $\mathcal{V}(\eta(\mathbf{x})) \neq \mathcal{V}(p_i(\mathbf{x}))$.*

Proof: If the above conditions are satisfied, we must have $p_i(\mathbf{x}) \neq \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$. Thus, the theorem holds. ■

In Fig. 4a and 4b, we present two examples for which the simple method is applicable. As shown in these examples, due to edge e , $\eta(\mathbf{x})$ contains effective variables $x_{\sigma_3 e}, x_{e \tau_2}$, which are absent in the unique form of $p_i(\mathbf{x})$ ($i = 1, 2, 3$). Thus, by Corollary VI.1, each unicast session can asymptotically achieve one half rate through PBNA. However, Corollary VI.1 doesn't subsume all possible networks for which PBNA is feasible. For instance, in Fig. 4c, we show a counter example, where $\mathcal{V}(\eta(\mathbf{x})) = \mathcal{V}(p_1(\mathbf{x}))$, and thus Corollary VI.1 is not applicable. Nevertheless, it is easy to verify the network satisfies the conditions of the Main Theorem, and thus PBNA is still feasible.

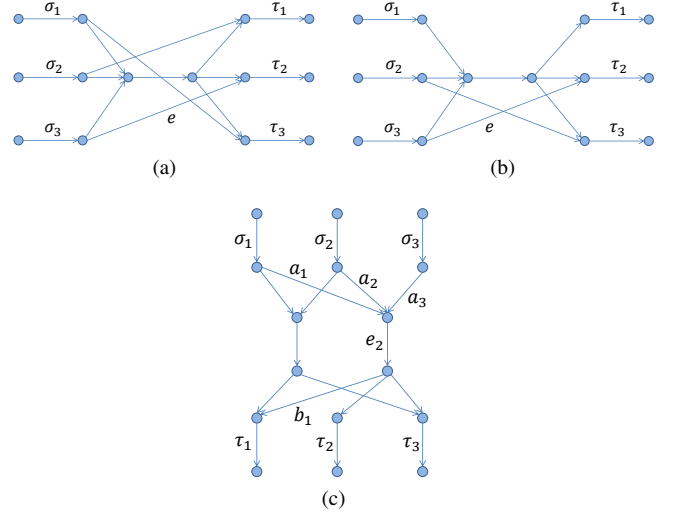


Fig. 4. Applying the simple method of Corollary VI.1 to networks: In the top two figures, due to edge e , $\eta(\mathbf{x})$ contains coding variables $x_{\sigma_3 e}, x_{e \tau_2}$, which are absent in the unique forms of $p_1(\mathbf{x}), p_2(\mathbf{x})$ and $p_3(\mathbf{x})$. Thus, the simple method applies to both networks. In contrast, the simple method fails in the third network due to $\mathcal{V}(\eta(\mathbf{x})) = \mathcal{V}(p_1(\mathbf{x}))$. However, PBNA is still feasible for this network according to the Main Theorem.

2) *Reducing \mathcal{S}' to \mathcal{S}'_i :* As shown in Section III-D, not all coupling relations $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$ are realizable due to the special properties of transfer functions. Indeed, using the two graph-related properties we introduce in Section V and a recent result of [17], we can reduce \mathcal{S}' to the minimal set \mathcal{S}'_i , which represents the set of all coupling relations $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$ that are realizable.

The proof consists of three steps. First, we use Linearization Property and a simple degree-counting technique to reduce \mathcal{S}' to the following set \mathcal{S}''_1 :

$$\mathcal{S}''_1 = \left\{ \frac{a_0 + a_1 \eta(\mathbf{x})}{b_0 + b_1 \eta(\mathbf{x})} \in \mathcal{S}' : a_0, a_1, b_0, b_1 \in \mathbb{F}_q \right\} \quad (23)$$

Note that \mathcal{S}''_1 only includes a finite number of rational functions. Next, we iterate through all possible configurations of a_0, a_1, b_0, b_1 , and utilize Linearization Property and Square-Term Property to further reduce \mathcal{S}''_1 to just four rational functions:

$$\mathcal{S}''_2 = \left\{ 1, \eta(\mathbf{x}), 1 + \eta(\mathbf{x}), \frac{\eta(\mathbf{x})}{1 + \eta(\mathbf{x})} \right\} \quad (24)$$

Finally, we use a recent result from [17] to rule out the fourth redundant rational function in \mathcal{S}''_2 , resulting in the minimal set \mathcal{S}'_i defined in the Main Theorem. The detailed proof is deferred to Appendix B.

3) *Necessity of the Feasibility Conditions:* We first show that the choices of precoding matrices are not limited to those defined in Eq. 7-9. In fact, given $\mathbf{A}, \mathbf{B}, \mathbf{C}$, we can always construct a precoding matrix \mathbf{V}_1 such that Eq. (5) is satisfied. The construction of \mathbf{V}_1 involves solving a system of linear equations defined on $\mathbb{F}_q(z)$:

$$\mathbf{r}(z)(z\mathbf{C} - \mathbf{B}\mathbf{A}) = \mathbf{0} \quad (25)$$

where $\mathbf{r}(z) = (r_1(z), \dots, r_{n+1}(z)) \in \mathbb{F}_{2^m}^{n+1}(z)$. Assume $\mathbf{r}_0(z)$ is a non-zero solution to Eq. (25). Substitute z with $\eta(\mathbf{x})$, and we have $\eta(\mathbf{x})\mathbf{r}_0(\eta(\mathbf{x}))\mathbf{C} = \mathbf{r}_0(\eta(\mathbf{x}))\mathbf{B}\mathbf{A}$. Finally, construct the following precoding matrix

$$\mathbf{V}_1^T = (\mathbf{r}_0^T(\eta(\mathbf{x}^1)) \quad \mathbf{r}_0^T(\eta(\mathbf{x}^2)) \quad \dots \quad \mathbf{r}_0^T(\eta(\mathbf{x}^{2n+1}))) \quad (26)$$

Apparently, \mathbf{V}_1 satisfies Eq. (5). Hence, each non-zero solution to Eq. (25) corresponds to a row of \mathbf{V}_1 satisfying Eq. (5). Conversely, it is straightforward to see that each row of \mathbf{V}_1 satisfying Eq. (5) corresponds to a solution to Eq. (25).

As an example, consider the case where $n = 2$ and $q = 4$. Let α be the primitive element of \mathbb{F}_4 such that $\alpha^3 = 1$ and $\alpha^2 + \alpha + 1 = 0$. Moreover, let $\mathbf{A} = \mathbf{I}_2$ and

$$\mathbf{C} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \\ \alpha^2 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} \alpha^2 & \alpha \\ 1 & 1 \\ 1 & \alpha \end{pmatrix}$$

It's easy to verify that $\mathbf{r}(z) = (\alpha^2 z^2 + \alpha, z + \alpha, z^2 + \alpha z + \alpha^2)$ satisfies Eq. (25). Thus, we substitute z with $\eta(\mathbf{x}^j)$ and construct $\mathbf{V}_1^T = (\mathbf{r}^T(\eta(\mathbf{x}^1)) \quad \mathbf{r}^T(\eta(\mathbf{x}^2)) \quad \dots \quad \mathbf{r}^T(\eta(\mathbf{x}^5)))$. Apparently, Eq. (5) is satisfied. From this example, we can see that given different $\mathbf{A}, \mathbf{B}, \mathbf{C}$, we can construct different precoding matrix \mathbf{V}_1 , and thus the choices of precoding matrices are not limited to those defined in Eq. (7)-(9). An interesting observation is that the above precoding matrix \mathbf{V}_1 is closely related to Eq. (7) through a transform equation: $\mathbf{V}_1 = \mathbf{V}_1^* \mathbf{F}$, where

$$\mathbf{F} = \begin{pmatrix} \alpha & \alpha & \alpha^2 \\ 0 & 1 & \alpha \\ \alpha^2 & 0 & 1 \end{pmatrix}$$

Actually, this observation can be generalized to the following Lemma.

Lemma VI.1. Any \mathbf{V}_1 satisfying Eq. (5) is related to \mathbf{V}_1^* through the following transform equation

$$\mathbf{V}_1 = \mathbf{G} \mathbf{V}_1^* \mathbf{F} \quad (27)$$

where \mathbf{V}_1^* is defined in Eq. (7), \mathbf{F} is an $(n+1) \times (n+1)$ matrix, and \mathbf{G} is a $(2n+1) \times (2n+1)$ diagonal matrix, with the (i, i) element being $f_i(\eta(\mathbf{x}^i))$, where $f_i(z)$ is an arbitrary non-zero rational function in $\mathbb{F}_q(z)$. Moreover, the $(n+1)$ th row of $\mathbf{F}\mathbf{C}$ and the 1st row of $\mathbf{F}\mathbf{B}\mathbf{A}$ are both zero vectors.

Proof: See Appendix B. ■

Using Lemma VI.1, we can prove that when PBNA is infeasible via the precoding matrices defined in Eq. (7)-(9) due to the presence of a coupling relation, it is infeasible via any precoding matrices which satisfy the alignment conditions. This implies that the conditions of the Main Theorem are also necessary for the feasibility of PBNA. We defer the detailed proof to Appendix B.

B. $\eta(\mathbf{x})$ Is Constant

For this case, we use a scheme that is slightly different from that of Section III. In this scheme, at each sender edge σ_i ($i = 1, 2, 3$), we encode source symbol X_i into two encoded

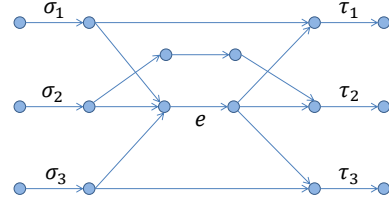


Fig. 5. An example where $\eta(\mathbf{x}) = 1$ and $p_i(\mathbf{x}) \neq 1$ for $i \in \{1, 2, 3\}$, and thus each unicast session can achieve one half rate in exactly two time slots due to Theorem IV.2.

symbols, which are then transmitted through the network in two time slots. The precoding matrices are defined as follows:

$$\mathbf{V}_1 = (\theta_1 \quad \theta_2)^T \quad (28)$$

$$\mathbf{V}_2 = \mathbf{M}_{13} \mathbf{M}_{23}^{-1} (\theta_1 \quad \theta_2)^T \quad (29)$$

$$\mathbf{V}_3 = \mathbf{M}_{12} \mathbf{M}_{32}^{-1} (\theta_1 \quad \theta_2)^T \quad (30)$$

where θ_1, θ_2 are arbitrary variables. Because \mathbf{T} is an identity matrix, it can be easily verified that at each receiver, the interferences are perfectly aligned, i.e.,

$$\mathbf{M}_{21} \mathbf{V}_2 = \mathbf{M}_{31} \mathbf{V}_3$$

$$\mathbf{M}_{32} \mathbf{V}_3 = \mathbf{M}_{12} \mathbf{V}_1$$

$$\mathbf{M}_{23} \mathbf{V}_2 = \mathbf{M}_{13} \mathbf{V}_1$$

Moreover, the rank conditions can be reformulated to similar forms to $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$.

Proof of Theorem IV.2: First, assume $p_i(\mathbf{x})$ is not constant. Using the above construction, the rank condition for the i th unicast session is equivalent to

$$\det \begin{pmatrix} \theta_1 & p_i(\mathbf{x}^1)\theta_1 \\ \theta_2 & p_i(\mathbf{x}^2)\theta_2 \end{pmatrix} = \theta_1\theta_2(p_i(\mathbf{x}^1) - p_i(\mathbf{x}^2)) \neq 0$$

Thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is feasible through PBNA. Conversely, if $p_i(\mathbf{x})$ is constant and thus $\mathbf{P}_i = \mathbf{I}_2$, the rank condition for the i th unicast session is violated, and thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is not feasible through PBNA. ■

In Fig. 5, we show an example of this case. One interesting observation is that in this example, each unicast session can achieve unit rate by using pure routing. Hence, PBNA doesn't provide any advantage over routing in terms of transmission rate.

C. Some $m_{ij}(\mathbf{x}) = 0$ ($i \neq j$)

In this case, since the number of interfering signals is reduced, at least one alignment conditions can be removed, and thus the restriction on \mathbf{V}_1 imposed by Eq. (5) vanishes. Therefore, we can choose \mathbf{V}_1 freely, and the feasibility conditions of PBNA can be greatly simplified. For example, assume $m_{21}(\mathbf{x}) = 0$ and all other transfer functions are non-zeros. Hence, the alignment condition for the first unicast session vanishes. Using a scheme similar to above, we set $\mathbf{V}_1 = (\theta_1 \quad \theta_2)^T$, $\mathbf{V}_2 = \mathbf{M}_{13} \mathbf{M}_{23}^{-1} (\theta_1 \quad \theta_2)^T$ and $\mathbf{V}_3 = \mathbf{M}_{12} \mathbf{M}_{32}^{-1} (\theta_1 \quad \theta_2)^T$, and thus the interferences at τ_2 and τ_3 are all perfectly aligned. It is easy to see that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is feasible through PBNA if and only if $p_i(\mathbf{x})$ is not constant

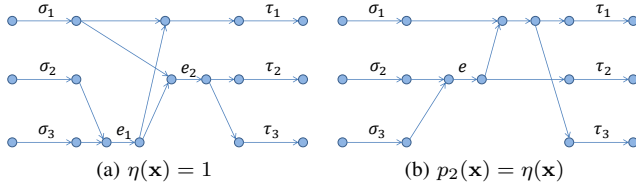


Fig. 6. Additional examples of coupling relations

for every $i = 1, 2, 3$. Using similar arguments, we can discuss other cases.

VII. INTERPRETATION AND CHECKING OF THE FEASIBILITY CONDITIONS OF PBNA

In this section, we first interpret the coupling conditions as defined in the Main Theorem in terms of the network structure. Then, based on these interpretation, we present a polynomial-time algorithm to check the feasibility of PBNA. We defer all the proofs to Appendix C.

In the following discussion, we use the following notations:

- $C_{ij,kl}$: The number of edges contained in the minimum cut which separates $\{\sigma_i, \sigma_j\}$ from $\{\tau_k, \tau_l\}$.
- $C_{e_1 e_2}$: The set of bottlenecks between two edges e_1 and e_2 . Here, we define an edge as a bottleneck between e_1 and e_2 if it forms a cut separating e_1 from e_2 . Moreover, we use C_{ij} to represent $C_{\sigma_i \tau_j}$.
- α_{ijk} : The last edge of the topological ordering of the edges in $C_{ij} \cap C_{ik}$. Here, given a set of edges \mathcal{T} , the topological ordering of \mathcal{T} is a rearrangement of the edges of \mathcal{T} , such that for any $e, e' \in \mathcal{T}$ and $\text{head}(e) = \text{tail}(e')$, e must precede e' in the ordering.
- β_{ijk} : The first edge of the topological ordering of the edges in $C_{jk} \cap C_{\alpha_{ijk}, \tau_k}$.
- $e_1 || e_2$: There is no directed path between two edges e_1 and e_2 .

A. Interpretation of Coupling Relations

Since the feasibility conditions of PBNA depend on whether $\eta(\mathbf{x})$ is constant, we first provide interpretation of the coupling relation $\eta(\mathbf{x}) = 1$, as shown below:

Theorem VII.1. $\eta(\mathbf{x}) = 1$ if and only if $\alpha_{213} = \alpha_{312}$ and $\beta_{213} = \beta_{312}$.

Consider the example shown in Fig. 3a. It is easy to see that in this example, $\alpha_{213} = \alpha_{312} = \beta_{213} = \beta_{312} = e$, and thus $\eta(\mathbf{x}) = 1$. In Fig. 6a, we show another example of $\eta(\mathbf{x}) = 1$. In this example, the last edges of $C_{21} \cap C_{23}$ and $C_{31} \cap C_{32}$ are both e_1 , and thus $\alpha_{213} = \alpha_{312} = e_1$. Moreover, the first edges of $C_{13} \cap C_{e_1 \tau_3}$ and $C_{12} \cap C_{e_1 \tau_2}$ are both e_2 , and thus $\beta_{213} = \beta_{312} = e_2$. Therefore, $\eta(\mathbf{x}) = 1$ according to Theorem VII.1.

The coupling relations $p_i(\mathbf{x}) = 1$ and $p_i(\mathbf{x}) = \eta(\mathbf{x})$ can be easily interpreted in terms of the minimum cut between two senders and two receivers, as shown below.

Theorem VII.2. The following statements hold:

- 1) $p_1(\mathbf{x}) = 1$ if and only if $C_{12,13} = 1$; $p_1(\mathbf{x}) = \eta(\mathbf{x})$ if and only if $C_{13,12} = 1$.
- 2) $p_2(\mathbf{x}) = 1$ if and only if $C_{12,23} = 1$; $p_2(\mathbf{x}) = \eta(\mathbf{x})$ if and only if $C_{23,12} = 1$.
- 3) $p_3(\mathbf{x}) = 1$ if and only if $C_{23,13} = 1$; $p_3(\mathbf{x}) = \eta(\mathbf{x})$ if and only if $C_{13,23} = 1$.

As shown previously in Fig. 3a, due to the single bottleneck e , the minimum cut separating all senders and receivers equals one, and thus $p_1(\mathbf{x}) = p_2(\mathbf{x}) = p_3(\mathbf{x}) = \eta(\mathbf{x}) = 1$. In contrast, in Fig. 6b, there is no single bottleneck, but because the minimum cut separating σ_2, σ_3 from τ_1, τ_2 contains only one edge e , the coupling relation $p_2(\mathbf{x}) = \eta(\mathbf{x})$ still holds.

Now we consider the remaining coupling relations, i.e., $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$, $p_2(\mathbf{x}) = 1 + \eta(\mathbf{x})$, and $p_3(\mathbf{x}) = 1 + \eta(\mathbf{x})$. It turns out that these coupling relations are more complicated than the previous ones, and cannot be easily interpreted in terms of cut conditions. In fact, each of them is characterized by two edges, as specified in Theorem VII.3. For instance, the coupling relation $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ is characterized by two edges α_{312} and α_{213} , which form a cut separating σ_1 from τ_1 .

Theorem VII.3. Assume $\eta(\mathbf{x})$ is not constant. The following statements hold:

- 1) $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ if and only if $\alpha_{312} \in C_{12}$, $\alpha_{213} \in C_{13}$, $\alpha_{312} || \alpha_{213}$, and $\{\alpha_{312}, \alpha_{213}\}$ forms a cut which separates σ_1 from τ_1 .
- 2) $p_2(\mathbf{x}) = 1 + \eta(\mathbf{x})$ if and only if $\alpha_{123} \in C_{23}$, $\alpha_{321} \in C_{21}$, $\alpha_{123} || \alpha_{321}$, and $\{\alpha_{123}, \alpha_{321}\}$ forms a cut which separates σ_2 from τ_2 .
- 3) $p_3(\mathbf{x}) = 1 + \eta(\mathbf{x})$ if and only if $\alpha_{231} \in C_{31}$, $\alpha_{132} \in C_{32}$, $\alpha_{231} || \alpha_{132}$, and $\{\alpha_{231}, \alpha_{132}\}$ forms a cut which separates σ_3 from τ_3 .

We have seen an example of coupling relation $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ in Fig. 3b. As shown in this figure, e_2 is the last edge of $C_{31} \cap C_{32}$, and e_1 the last edge of $C_{21} \cap C_{23}$. Thus $e_2 = \alpha_{312}$ and $e_1 = \alpha_{213}$. Moreover, $e_2 \in C_{12}$, $e_1 \in C_{13}$, $e_1 || e_2$, and $\{e_1, e_2\}$ forms a cut separating σ_1 from τ_1 .

B. Checking the Feasibility Conditions of PBNA

We assume G' is stored as an adjacency list, i.e., for each node $v \in V'$, we associate it with the set of its incoming edges and the set of its outgoing edges. Moreover, we assume all the edges in G' have been arranged in topological order.

The checking process consists of the following steps: 1) Check if $\eta(\mathbf{x}) = 1$; 2) if $\eta(\mathbf{x}) = 1$, check the conditions of Theorem IV.2; 3) otherwise, check the conditions of the Main Theorem. In the following discussion, we present the building blocks involved in these steps.

1) *Calculating $C_{ee'}$* : We use Algorithm 1 to calculate the set of bottlenecks $C_{ee'}$ which separates e from e' . The algorithm consists of two steps: 1) Lines 1-3 are used to calculate the set of edges traversed by the paths in $P_{ee'}$, denoted by $E_{ee'}$. Note that in the reverse BFS algorithm, we start from e' and move upwards by following the incoming edges associated with each node. 2) Lines 4-11 are used to calculate $C_{ee'}$. In this step, we iterate through each edge $e \in E_{ee'}$ in the topological order. In each iteration, we calculate C , which forms a cut

Algorithm 1: Calculate $\mathcal{C}_{ee'}$

```

1 Use BFS algorithm to calculate the set of edges reachable from
  e, denoted by  $E_1$ ;
2 Use reverse BFS algorithm to calculate the set of edges which
  is connected to  $e'$ , denoted by  $E_2$ ;
3  $E_{ee'} \leftarrow E_1 \cap E_2$ ;
4  $\mathcal{C}_{ee'} \leftarrow \{\sigma_i\}$ ,  $\mathcal{C} \leftarrow \{\sigma_i\}$ ;
5 for each  $e \in E_{ee'}$  in the topological order do
6    $\mathcal{C} \leftarrow \mathcal{C} - \{e\}$ ;
7   for each  $e'$  such that  $\text{head}(e) = \text{tail}(e')$  do
8     if  $e' \in E_{ee'}$  then  $\mathcal{C} \leftarrow \mathcal{C} \cup \{e'\}$ ;
9   end
10 if  $\mathcal{C}$  contains one edge then  $\mathcal{C}_{ee'} \leftarrow \mathcal{C}_{ee'} \cup \mathcal{C}$ ;
11 end

```

Algorithm 2: Check if $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$

```

1  $\alpha_{312} \leftarrow$  the last edge of  $\mathcal{C}_{31} \cap \mathcal{C}_{32}$ ;
2  $\alpha_{213} \leftarrow$  the last edge of  $\mathcal{C}_{21} \cap \mathcal{C}_{23}$ ;
3 if  $\alpha_{312} \notin \mathcal{C}_{12}$  or  $\alpha_{213} \notin \mathcal{C}_{13}$  then return false;
4 Use BFS algorithm to check whether  $\alpha_{312}$  is connected with
   $\alpha_{213}$  by a directed path;
5 if  $\alpha_{312}$  is connected with  $\alpha_{213}$  then return false;
6 Let  $G_1$  denote the subgraph of  $G'$  induced by
   $E' - \{\alpha_{312}, \alpha_{213}\}$ ;
7 Use BFS algorithm to check whether  $\tau_1$  is connected to  $\sigma_1$  in
   $G_1$ ;
8 if  $\tau_1$  is connected to  $\sigma_1$  in  $G_1$  then return false;
9 else return true;

```

separating e from e' . If \mathcal{C} contains only one edge, we then incorporate \mathcal{C} into $\mathcal{C}_{ee'}$. The running time of the algorithm is $O(h|E'|)$, where h is the maximum in-degree of nodes in G' .

2) *Checking if $\eta(\mathbf{x}) = 1$:* Using algorithm 1 and Theorem VII.1, we can easily check whether this coupling relation holds. First, we calculate $\mathcal{C}_{31} \cap \mathcal{C}_{32}$, $\mathcal{C}_{21} \cap \mathcal{C}_{23}$, from which we get the two edges α_{312} and α_{213} . Then, we calculate $\mathcal{C}_{12} \cap \mathcal{C}_{\alpha_{312}, \tau_2}$, $\mathcal{C}_{13} \cap \mathcal{C}_{\alpha_{213}, \tau_3}$, from which we get β_{312} and β_{213} . Finally, we use Theorem VII.1 to check if $\eta(\mathbf{x}) = 1$ by checking whether $\alpha_{312} = \beta_{312}$ and $\alpha_{213} = \beta_{213}$.

3) *Checking if $p_i(\mathbf{x}) = 1$ or $p_i(\mathbf{x}) = \eta(\mathbf{x})$:* Due to Theorem VII.2, we use Ford-Fulkerson Algorithm to check these coupling relations. For example, in order to check whether $p_1(\mathbf{x}) = 1$, we add a super sender node s' , which is connected to s'_1 and s'_2 via two directed edges of capacity one, and a super receiver node d' , to which d'_1 and d'_3 are connected via two directed edges of capacity one. We then use Ford-Fulkerson Algorithm to calculate the maximum flow from s' to d' , which is identical to $C_{12,13}$. Thus, by checking whether $C_{12,13} = 1$, we can identify whether $p_1(\mathbf{x}) = 1$. Similarly, we can check other coupling relations.

4) *Checking if $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ or $p_2(\mathbf{x}), p_3(\mathbf{x}) = 1 + \eta(\mathbf{x})$:* We use Algorithm 2 to check if $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$. The other two coupling relations can be checked similarly. Note that Line 4 consists of two steps: First, we start from α_{312} and use BFS to check if α_{213} is reachable from α_{312} ; then we start from α_{213} and use BFS to check if α_{312} is reachable from α_{213} . The running time of the algorithm is $O(h|E'|)$.

VIII. CONCLUSION AND FUTURE WORK

In this paper, we consider the problem of network coding for the SISO network with three unicast sessions. We introduce how to apply precoding-based interference alignment [14] to this network setting. We show that due to network topology, there might be dependency relations, called coupling relations, between different transfer functions, which can potentially make PBNA infeasible. Using two graph-related properties and a recent result from [17], we identify the minimal set of coupling relations that are realizable in networks. Moreover, we show that each of these coupling relations has a unique interpretation in terms of network topology. Based on these interpretations, we present a polynomial-time algorithm to check the feasibility of PBNA.

Clearly, the scenario considered in this paper (SISO network with three unicast sessions) is the simplest meaningful scenario in this setting but there still many problems that remain to be solved regarding applying interference alignment techniques to the network setting. One important problem is the complexity of PBNA, which arises in two aspects, i.e., precoding matrix and field size, and is inherent in the framework of PBNA. Another important issue is how much benefits interference alignment provides compared with routing schemes. One direction for future work is to apply other alignment techniques with lower complexity to the network setting. The extensions to other networks scenarios beyond SISO with more than three unicast sessions are highly non-trivial. Other extensions of the current framework to more practical scenarios include networks with delays and errors.

APPENDIX A

PROOFS OF GRAPH-RELATED PROPERTIES

A. Linearization Property and Square-Term Property

The following lemma plays an important role in the proof of Linearization Property and the interpretation of the coupled relations, $p_i(\mathbf{x}) = 1$ and $p_i(\mathbf{x}) = \eta(\mathbf{x})$. The basic idea of this lemma is that we can multicast two symbols from two senders to two receivers via network coding if and only if the minimum cut separating the senders from the receivers is greater than one.

Lemma A.1. $m_{ab}(\mathbf{x})m_{pq}(\mathbf{x}) \neq m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})$ if and only if there is disjoint path pair $(P_1, P_2) \in \mathcal{P}_{ab} \times \mathcal{P}_{pq}$ or $(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}$.

Proof: We add a super sender s and connect it to s'_a and s'_p via two edges of unit capacity, and a super receiver d , to which we connect d'_b and d'_q via two edges of unit capacity. Thus, the transfer matrix at d is

$$\mathbf{M} = \begin{pmatrix} m_{ab}(\mathbf{x}) & m_{aq}(\mathbf{x}) \\ m_{pb}(\mathbf{x}) & m_{pq}(\mathbf{x}) \end{pmatrix}$$

It is easy to see $\det(\mathbf{M}) = m_{ab}(\mathbf{x})m_{pq}(\mathbf{x}) - m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})$. Hence, we can multicast two symbols from s to d , i.e., $\det(\mathbf{M}) \neq 0$, if and only if the minimum cut separating s from d is at least two, or equivalently there is a disjoint path pair $(P_1, P_2) \in \mathcal{P}_{ab} \times \mathcal{P}_{pq}$ or $(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}$. ■

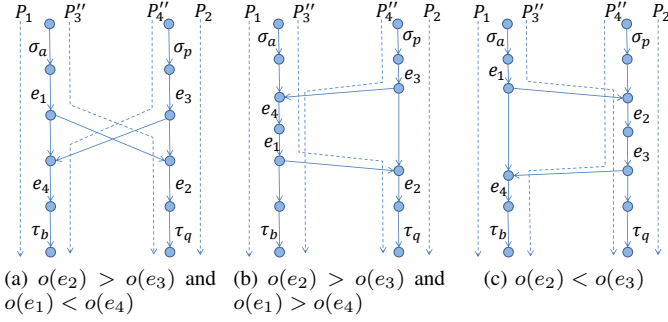


Fig. 7. The construction of H (in the proof of the Linearization Property) enabled by Lemma A.1 (P_1 is disjoint with P_2)

The proof of Linearization Property involves finding a subgraph H such that some coding variable appears exclusively in the denominator or numerator of $h(\mathbf{x}_H)$, i.e., $h(\mathbf{x}_H)$ restricted to H . In fact, due to Lemma A.1, such subgraph H always exists, if $h(\mathbf{x})$ is not constant.

Proof of Linearization Property: In this proof, given a path P , let $P[e : e']$ denote the path segment of P between two edges e and e' , including e, e' . We arrange the edges of G' in topological order, and for $e \in E'$, let $o(e)$ denote e 's position in this ordering. Moreover, denote $h_1(\mathbf{x}) = m_{ab}(\mathbf{x})m_{pq}(\mathbf{x})$, $h_2(\mathbf{x}) = m_{aq}(\mathbf{x})m_{pb}(\mathbf{x})$ and $d(\mathbf{x}) = \gcd(h_1(\mathbf{x}), h_2(\mathbf{x}))$. Let $s_1(\mathbf{x}) = \frac{h_1(\mathbf{x})}{d(\mathbf{x})}$ and $s_2(\mathbf{x}) = \frac{h_2(\mathbf{x})}{d(\mathbf{x})}$. Hence $\gcd(s_1(\mathbf{x}), s_2(\mathbf{x})) = 1$. It follows $u(\mathbf{x}) = cs_1(\mathbf{x}), v(\mathbf{x}) = cs_2(\mathbf{x})$, where c is a non-zero constant in \mathbb{F}_m . By Lemma A.1, there exists disjoint path pair $(P_1, P_2) \in \mathcal{P}_{ab} \times \mathcal{P}_{pq}$ or $(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}$. Now we consider the first case.

We arbitrarily select another path pair $(P'_3, P'_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}$. Since P_1, P'_3 both originate at σ_a , and P_2, P'_3 both terminate at τ_q , there exist $e_1 \in P_1 \cap P'_3$ and $e_2 \in P_2 \cap P'_3$ such that the path segment along P'_3 between e_1 and e_2 is disjoint with $P_1 \cup P_2$. Similarly, there exist $e_3 \in P_2 \cap P'_4$ and $e_4 \in P_1 \cap P'_4$ such that the path segment between e_3 and e_4 along P'_4 is disjoint with $P_1 \cup P_2$. Construct the following two paths: $P'_3 = P_1[\sigma_a : e_1] \cup P'_3[e_1 : e_2] \cup P_2[e_2 : \tau_q]$ and $P'_4 = P_2[\sigma_p : e_3] \cup P'_4[e_3 : e_4] \cup P_1[e_4 : \tau_b]$ (see Fig. 7). Let H denote the subgraph of G' induced by $P_1 \cup P_2 \cup P'_3 \cup P'_4$.

We then prove that the theorem holds for H . If $o(e_2) > o(e_3)$ (Fig. 7a and 7b), the variables along $P_2[e_3 : e_2]$ are absent in $h_2(\mathbf{x}_H)$. We then arbitrarily select a variable $x_{ee'}$ from $P_2[e_3 : e_2]$, and write $h_1(\mathbf{x}_H)$ as $f(\mathbf{x}'_H)x_{ee'} + g(\mathbf{x}'_H)$, where \mathbf{x}'_H includes all the variables in \mathbf{x}_H other than $x_{ee'}$. Meanwhile, $h_2(\mathbf{x}_H)$ can be written as $h_2(\mathbf{x}'_H)$. Clearly, $x_{ee'}$ will not show up in $d(\mathbf{x}_H)$ and thus it can also be written as $d(\mathbf{x}'_H)$. We then find values for \mathbf{x}'_H , denoted by \mathbf{r} , such that $f(\mathbf{r})h_2(\mathbf{r})d(\mathbf{r}) \neq 0$. Finally, denote $c_0 = cg(\mathbf{r})d^{-1}(\mathbf{r})$, $c_1 = cf(\mathbf{r})d^{-1}(\mathbf{r})$ and $c_2 = ch_2(\mathbf{r})d^{-1}(\mathbf{r})$ and the theorem holds. On the other hand, if $o(e_2) < o(e_3)$ (see Fig. 7c), the variables along $P_1[e_1 : e_4]$ are absent in $h_2(\mathbf{x}_H)$. We then select a variable $x_{ee'}$ from $P_1[e_1 : e_4]$. Similar to above, it's easy to see that $u(\mathbf{x})$ and $v(\mathbf{x})$ can be transformed into $c_1x_{ee'} + c_0$ and c_2 respectively.

For the case where $(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb}$ is a disjoint path

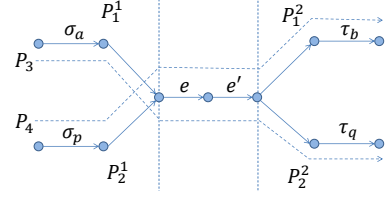


Fig. 8. Illustration of Square-Term Property. A term with $x_{ee'}^2$, introduced by (P_1, P_2) in the numerator of $h(\mathbf{x})$ equals another term introduced by (P_3, P_4) in the denominator of $h(\mathbf{x})$.

pair, we can show that $u(\mathbf{x})$ and $v(\mathbf{x})$ can be transformed into c_2 and $c_1x_{ee'} + c_0$ respectively. ■

The basic idea of Square-Term Property is to construct a one-to-one mapping between the square terms in the numerator of $h(\mathbf{x})$ and those in the denominator of $h(\mathbf{x})$.

Proof of Square-Term Property: First, we define two sets $\mathcal{Q}_1 = \{(P_1, P_2) \in \mathcal{P}_{ab} \times \mathcal{P}_{pq} : x_{ee'}^2 \mid t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x})\}$ and $\mathcal{Q}_2 = \{(P_3, P_4) \in \mathcal{P}_{aq} \times \mathcal{P}_{pb} : x_{ee'}^2 \mid t_{P_3}(\mathbf{x})t_{P_4}(\mathbf{x})\}$. Consider a path pair $(P_1, P_2) \in \mathcal{Q}_1$. Since the degree of $x_{ee'}$ in $t_{P_1}(\mathbf{x})$ and $t_{P_2}(\mathbf{x})$ is at most one, we must have $x_{ee'} \mid t_{P_1}(\mathbf{x})$ and $x_{ee'} \mid t_{P_2}(\mathbf{x})$. Thus $e, e' \in P_1 \cap P_2$. Let P_1^1, P_1^2 be the parts of P_1 before e and after e' respectively. Similarly, define P_2^1 and P_2^2 . Then construct two new paths: $P_3 = P_1^1 \cup \{e, e'\} \cup P_2^2$ and $P_4 = P_2^1 \cup \{e, e'\} \cup P_1^2$ (see Fig. 8). Clearly, $t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x}) = t_{P_3}(\mathbf{x})t_{P_4}(\mathbf{x})$, and thus $(P_3, P_4) \in \mathcal{Q}_2$. The above method establishes a one-to-one mapping $\phi : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$, such that for $\phi((P_1, P_2)) = (P_3, P_4)$, $t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x}) = t_{P_3}(\mathbf{x})t_{P_4}(\mathbf{x})$. Hence, $f_1(\mathbf{x}) = \frac{1}{x_{ee'}^2} \sum_{(P_1, P_2) \in \mathcal{Q}_1} t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x}) = \frac{1}{x_{ee'}^2} \sum_{(P_3, P_4) \in \mathcal{Q}_2} t_{P_3}(\mathbf{x})t_{P_4}(\mathbf{x}) = f_2(\mathbf{x})$. ■

B. Other Graph-Related Properties

In this section, we present other graph-related properties, which reveal more microscopic structures of transfer functions, and are to be used in the proofs of Theorems VII.1 and VII.3. Before proceeding, we first extend the concept of transfer function to any two edges $e, e' \in E'$, i.e., $m_{ee'}(\mathbf{x}) = \sum_{P \in \mathcal{P}_{ee'}} t_P(\mathbf{x})$, where $\mathcal{P}_{ee'}$ is the set of paths from e to e' .

The following lemma states that any transfer function $m_{ee'}(\mathbf{x})$ is fully determined by the two edges e, e' .

Lemma A.2. Consider two transfer functions $m_{e_1e_2}(\mathbf{x})$ and $m_{e_3e_4}(\mathbf{x})$. Then $m_{e_1e_2}(\mathbf{x}) = m_{e_3e_4}(\mathbf{x})$ if and only if $e_1 = e_3$ and $e_2 = e_4$.

Proof: Apparently, the “if” part holds trivially. Now assume $e_1 \neq e_3$ or $e_2 \neq e_4$. Then, there must be some edge which appears exclusively in $\mathcal{P}_{e_1e_2}$ or $\mathcal{P}_{e_3e_4}$, implying $m_{e_1e_2}(\mathbf{x}) \neq m_{e_3e_4}(\mathbf{x})$. Thus, the lemma holds. ■

The following result was first proved by Han et al. [17]. It states that each transfer function $m_{ee'}(\mathbf{x})$ can be uniquely factorized into a product of irreducible polynomials according to the bottlenecks between e and e' .

Lemma A.3. We arrange the bottlenecks in $\mathcal{C}_{ee'}$ in topological order: e_1, e_2, \dots, e_k , such that $e = e_1, e' = e_k$. Then, $m_{ee'}(\mathbf{x})$ can be factorized as $m_{ee'}(\mathbf{x}) = \prod_{i=1}^{k-1} m_{e_ie_{i+1}}(\mathbf{x})$, where $m_{e_ie_{i+1}}(\mathbf{x})$ is an irreducible polynomial.

In addition, as shown below, any transfer function $m_{ee'}(\mathbf{x})$ can be partitioned into a summation of products of transfer functions according to a cut between e and e' .

Lemma A.4. Assume $\mathcal{U} = \{e_1, e_2, \dots, e_k\}$ is a cut which separates e from e' . If $e_i \parallel e_j$ for $e_i \neq e_j \in \mathcal{U}$, we have $m_{ee'}(\mathbf{x}) = \sum_{i=1}^k m_{ee_i}(\mathbf{x})m_{e_i e'}(\mathbf{x})$. Otherwise, the above equality doesn't hold.

Proof: For $e_i \in \mathcal{U}$, let $\mathcal{P}_{ee'}^i$ denote the set of paths in $\mathcal{P}_{ee'}$ which pass through e_i . Because $e_i \parallel e_j$ for $e_i \neq e_j \in \mathcal{U}$, $\mathcal{P}_{ee'}^i$ is disjoint with $\mathcal{P}_{ee'}^j$. Hence, $m_{ee'}(\mathbf{x}) = \sum_{i=1}^k \sum_{P \in \mathcal{P}_{ee'}^i} t_P(\mathbf{x})$. Note that $m_{ee_i}(\mathbf{x})m_{e_i e'}(\mathbf{x}) = \sum_{(P_1, P_2) \in \mathcal{P}_{ee_i} \times \mathcal{P}_{e_i e'}} t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x})$. Moreover, each monomial $t_P(\mathbf{x})$ in $m_{ee'}(\mathbf{x})$ corresponds to a monomial $t_{P_1}(\mathbf{x})t_{P_2}(\mathbf{x})$ in $m_{ee_i}(\mathbf{x})m_{e_i e'}(\mathbf{x})$. Hence, $m_{ee_i}(\mathbf{x})m_{e_i e'}(\mathbf{x}) = \sum_{P \in \mathcal{P}_{ee'}^i} t_P(\mathbf{x})$, and the lemma holds. On the other hand, if some e_i is upstream of e_j , $\mathcal{P}_{ee'}^i \cap \mathcal{P}_{ee'}^j \neq \emptyset$, and thus $m_{ee'}(\mathbf{x}) \neq \sum_{i=1}^k \sum_{P \in \mathcal{P}_{ee'}^i} t_P(\mathbf{x})$, indicating that the lemma doesn't hold. ■

APPENDIX B

PROOFS OF FEASIBILITY CONDITIONS OF PBNA

A. Reducing \mathcal{S}' to \mathcal{S}'_i

In order to utilize the degree-counting technique, we use the following lemma. Basically, it allows us to reformulate each $\frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$ to its unique form $\frac{\alpha(\mathbf{x})}{\beta(\mathbf{x})}$, such that we can compare the degrees of a coding variable in $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ with its degrees in the numerator and denominator of $p_i(\mathbf{x})$ respectively.

Lemma B.1. Let \mathbb{F} be a field. z is a variable and $\mathbf{y} = (y_1, y_2, \dots, y_k)$ is a vector of variables. Consider four non-zero polynomials $f(z), g(z) \in \mathbb{F}[z]$ and $s(\mathbf{y}), t(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$, such that $\gcd(f(z), g(z)) = 1$ and $\gcd(s(\mathbf{y}), t(\mathbf{y})) = 1$. Denote $d = \max\{d_f, d_g\}$. Define two polynomials in $\mathbb{F}[\mathbf{y}]$: $\alpha(\mathbf{y}) = f(\frac{s(\mathbf{y})}{t(\mathbf{y})})t^d(\mathbf{y})$ and $\beta(\mathbf{y}) = g(\frac{s(\mathbf{y})}{t(\mathbf{y})})t^d(\mathbf{y})$. Then $\gcd(\alpha(\mathbf{y}), \beta(\mathbf{y})) = 1$.

Proof: See Appendix D. ■

We use the following three steps to reduce \mathcal{S}' to \mathcal{S}'_i .

Step 1: $\mathcal{S}' \Rightarrow \mathcal{S}'_1$. Assume $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}'$. We will prove that $d = \max\{d_f, d_g\} = 1$. Let $p_i(\mathbf{x}) = \frac{u(\mathbf{x})}{v(\mathbf{x})}$, $\eta(\mathbf{x}) = \frac{s(\mathbf{x})}{t(\mathbf{x})}$ denote the unique forms of $p_i(\mathbf{x})$ and $\eta(\mathbf{x})$ respectively. Without loss of generality, let $f(z) = \sum_{j=0}^k a_j z^j$, $g(z) = \sum_{j=0}^l b_j z^j$ where $a_k b_l \neq 0$. We first consider the case where $l \leq k$ and thus $d = k$. Define the following two polynomials:

$$\alpha(\mathbf{x}) = f(\eta(\mathbf{x}))t^k(\mathbf{x}) = \sum_{j=0}^k a_j t^{k-j}(\mathbf{x})s^j(\mathbf{x})$$

$$\beta(\mathbf{x}) = g(\eta(\mathbf{x}))t^k(\mathbf{x}) = \sum_{j=0}^l b_j t^{k-j}(\mathbf{x})s^j(\mathbf{x})$$

Due to Lemma B.1, we have $\alpha(\mathbf{x}) = cu(\mathbf{x}), \beta(\mathbf{x}) = cv(\mathbf{x})$, where c is a non-zero constant in \mathbb{F}_q . Moreover, according to Linearization Property, we assign values to \mathbf{x} other than a coding variable $x_{ee'}$ such that $u(\mathbf{x})$ and $v(\mathbf{x})$ are transformed

into:

$$\begin{aligned} u(x_{ee'}) &= c_1 x_{ee'} + c_0 & v(x_{ee'}) &= c_2 \\ \text{or } u(x_{ee'}) &= c_2 & v(x_{ee'}) &= c_1 x_{ee'} + c_0 \end{aligned}$$

where $c_0, c_1, c_2 \in \mathbb{F}_q$ and $c_1 c_2 \neq 0$. We only consider the first case. The proof for the other case is similar. In this case, $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are transformed into $\alpha(x_{ee'}) = cc_1 x_{ee'} + cc_0$ and $\beta(x_{ee'}) = cc_2$ respectively.

By contradiction, assume $d \geq 2$. We first consider the case where $l \leq k$ and thus $d = k$. In this case, we have

$$\alpha(x_{ee'}) = \sum_{j=0}^k a_j t^{k-j}(x_{ee'}) s^j(x_{ee'}) = cc_1 x_{ee'} + cc_0$$

$$\beta(x_{ee'}) = \sum_{j=0}^l b_j t^{k-j}(x_{ee'}) s^j(x_{ee'}) = cc_2$$

Assume $s(x_{ee'}) = \sum_{j=0}^r s_j x_{ee'}^j$ and $t(x_{ee'}) = \sum_{j=0}^{r'} t_j x_{ee'}^j$, where $s_r, t_{r'} \neq 0$. Thus $\max\{r, r'\} \geq 1$. Note that the degree of $x_{ee'}$ in $t^{k-j}(x_{ee'}) s^j(x_{ee'})$ is $kr' + j(r - r')$. We consider the following two cases:

Case I: $r \neq r'$. If $r > r'$, $d_\alpha = kr \geq 2$, contradicting that $d_\alpha = 1$. Now assume $r < r'$. Let l_1 and l_2 be the minimum exponents of z in $f(z)$ and $g(z)$ respectively. It follows that $d_\alpha = kr' - l_1(r' - r) = 1$ and $d_\beta = kr' - l_2(r' - r) = 0$. Clearly, $l_2 > 0$ due to $d_\beta = 0$. If $r > 0$, $kr' - l_2(r' - r) > kr' - l_2 r' \geq 0$, contradicting $d_\beta = 0$. Hence, $r = 0$, and $l_2 = k$ due to $d_\beta = 0$. Meanwhile, $d_\alpha = (k - l_1)r' = 1$, which implies that $l_1 = k - 1$ and $r' = 1$. Thus, z^{k-1} is a common divisor of $f(z)$ and $g(z)$, contradicting $\gcd(f(z), g(z)) = 1$.

Case II: $r = r'$. Since $d_\alpha = 1$ and $d_\beta = 0$, all the terms in $\alpha(x_{ee'})$ and $\beta(x_{ee'})$ containing $x_{ee'}^{kr}$ must be cancelled out, implying that

$$\begin{aligned} \sum_{j=0}^k a_j t_r^{k-j} s_r^j &= t_r^k \sum_{j=0}^k a_j \left(\frac{s_r}{t_r}\right)^j = t_r^k f\left(\frac{s_r}{t_r}\right) = 0 \\ \sum_{j=0}^l b_j t_r^{k-j} s_r^j &= t_r^k \sum_{j=0}^l b_j \left(\frac{s_r}{t_r}\right)^j = t_r^k g\left(\frac{s_r}{t_r}\right) = 0 \end{aligned}$$

Hence $z - \frac{s_r}{t_r}$ is a common divisor of $f(z)$ and $g(z)$, contradicting $\gcd(f(z), g(z)) = 1$.

Therefore, we have proved $d = 1$ when $l \leq k$. Using similar technique, we can prove that $d = 1$ when $l \geq k$. This implies that $\frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$ can only be of the form $\frac{a_0 + a_1 \eta(\mathbf{x})}{b_0 + b_1 \eta(\mathbf{x})}$. Hence, we have reduced \mathcal{S}' to \mathcal{S}'_1 .

Step 2: $\mathcal{S}'_1 \Rightarrow \mathcal{S}'_2$. We consider the coupling relation $p_1(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$. The coupling relations $p_2(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$ and $p_3(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$ can be dealt with similarly. Define $q_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{p_1(\mathbf{x})} = \frac{m_{11}(\mathbf{x})m_{23}(\mathbf{x})}{m_{13}(\mathbf{x})m_{21}(\mathbf{x})}$. Assume the characteristic of \mathbb{F}_q is p . Given an integer m , let m_p denote the remainder of m divided by p . Since \mathcal{S}'_1 only consists of a finite number of rational functions, we iterate all possible configurations of a_0, a_1, b_0, b_1 as follows:

Case I: $\frac{f(z)}{g(z)} = \frac{a_0 + a_1 z}{b_0 + b_1 z}$, where $a_1 a_0 b_1 b_0 \neq 0$, and $a_0 b_1 \neq a_1 b_0$. For this case, we have $p_1(x_{ee'}) = \frac{a_0 + a_1 p_1(x_{ee'}) q_1(x_{ee'})}{b_0 + b_1 p_1(x_{ee'}) q_1(x_{ee'})}$. It immediately follows

$$q_1(x_{ee'}) = \frac{a_0 c_2^2 - b_0 c_0 c_2 - b_0 c_1 c_2 x_{ee'}}{b_1 c_1^2 x_{ee'}^2 + (2p b_1 c_0 c_1 - a_1 c_1 c_2) x_{ee'} + b_1 c_0^2 - a_1 c_0 c_2}$$

Let $u_1(x_{ee'}), v_1(x_{ee'})$ denote the numerator and denominator of the above equation respectively. Assume $u_1(x_{ee'}) \mid v_1(x_{ee'})$ and thus $x_{ee'} = \frac{a_0 c_2 - b_0 c_0}{b_0 c_1}$ is a solution to $v_1(x_{ee'}) = 0$. However, $v_1(\frac{a_0 c_2 - b_0 c_0}{b_0 c_1}) = \frac{a_0 c_2^2}{b_0^2} (a_0 b_1 - a_1 b_0) \neq 0$. Hence, $u_1(x_{ee'}) \nmid v_1(x_{ee'})$. Thus, by the definition of $q_1(\mathbf{x})$ and Square-Term Property, $x_{ee'}^2$ must appear in $u_1(x_{ee'})$, which contradicts the formulation of $u_1(x_{ee'})$.

Case II: $\frac{f(z)}{g(z)} = \frac{a_0 + a_1 z}{b_1 z}$, where $a_0 a_1 b_0 \neq 0$. Similar to Case I, we can derive

$$q_1(x_{ee'}) = \frac{a_0 c_2^2}{b_1 c_1^2 x_{ee'}^2 + (2_p b_1 c_0 c_1 - a_1 c_1 c_2) x_{ee'} + b_1 c_0^2 - a_1 c_0 c_2}$$

which contradicts Square-Term Property.

Case III: $\frac{f(z)}{g(z)} = \frac{a_1 z}{b_0 + b_1 z}$, where $a_1 b_0 b_1 \neq 0$. Thus $\frac{1}{p_1(\mathbf{x})} = \frac{b_0}{a_1} \frac{1}{\eta(\mathbf{x})} + \frac{b_1}{a_1}$. Since the coefficient of each monomial in the denominators and numerators of $p_1(\mathbf{x})$ and $\eta(\mathbf{x})$ equals one, it follows $\frac{a_0}{b_1} = \frac{b_1}{a_1} = 1$. This indicates that $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{\eta(\mathbf{x}) + 1}$.

Case IV: $\frac{f(z)}{g(z)} = \frac{a_0}{b_0 + b_1 z}$, where $a_0 b_0 b_1 \neq 0$. It follows that

$$q_1(x_{ee'}) = \frac{a_0 c_2^2 - b_0 c_0 c_2 - b_0 c_1 c_2 x_{ee'}}{b_1 c_0^2 + 2_p b_1 c_0 c_1 x_{ee'} + b_1 c_1^2 x_{ee'}^2}$$

Similar to Case I, this also contradicts Square-Term Property.

Case V: $\frac{f(z)}{g(z)} = \frac{a_0}{z}$, where $a_0 \neq 0$. Hence, $q_1(x_{ee'}) = \frac{a_0 c_2^2}{c_1^2 x_{ee'}^2 + 2_p c_0 c_1 x_{ee'} + c_0^2}$, contradicting Square-Term Property.

Case VI: $\frac{f(z)}{g(z)} = a_0 + a_1 z$, where $a_0 a_1 \neq 0$. Thus, it follows $p_1(\mathbf{x}) = a_0 + a_1 \eta(\mathbf{x})$. Similar to Case III, $a_1 = a_0 = 1$, implying that $p_1(\mathbf{x}) = 1 + \eta(\mathbf{x})$.

Case VII: $\frac{f(z)}{g(z)} = a_1 z$, where $a_1 \neq 0$. Similar to Case III, $a_1 = 1$ and hence $p_1(\mathbf{x}) = \eta(\mathbf{x})$.

Therefore, we have proved that $\frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))}$ can only take the form of the four rational functions in \mathcal{S}_2'' . Thus, we have reduced \mathcal{S}_1'' to \mathcal{S}_2'' .

Step 3: $\mathcal{S}_2'' \Rightarrow \mathcal{S}_i'$. We note that in Proposition 3 of [17], it was proved that $p_1(\mathbf{x}) \neq 1 + \eta(\mathbf{x})$, $p_2(\mathbf{x}) \neq \frac{\eta(\mathbf{x})}{1 + \eta(\mathbf{x})}$ and $p_3(\mathbf{x}) \neq \frac{\eta(\mathbf{x})}{1 + \eta(\mathbf{x})}$. Combined with the above results, we have reduced \mathcal{S}_2'' to \mathcal{S}_i' .

In summary, according to Theorem III.1, if the conditions of the Main Theorem are satisfied, the three unicast sessions can asymptotically achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ through PBNA.

B. Necessity of the Feasibility Conditions

As shown previously, each row of \mathbf{V}_1 satisfying the alignment conditions corresponds to a non-zero solution to Eq. (25). The following lemma reveals that any non-zero solution to Eq. (25) is linearly dependent on the particular vector $(1, z, z^2, \dots, z^n)$, which forms each row of the precoding matrix \mathbf{V}_1^* .

Lemma B.2. *Eq. (25) has a non-zero solution in $\mathbb{F}_{2^m}^{n+1}[z]$ in the form of $\mathbf{r}(z) = (1, z, z^2, \dots, z^n)\mathbf{F}$, where \mathbf{F} is an $(n+1) \times (n+1)$ matrix in \mathbb{F}_{2^m} . Moreover, any solution to Eq. (25) is linearly dependent on $(1, z, \dots, z^n)\mathbf{F}$.*

Proof: Denote $\mathbf{D} = \mathbf{B}\mathbf{A}$. First, we will prove that $\text{rank}(z\mathbf{C} - \mathbf{D}) = n$. Let \mathbf{c}_i and \mathbf{d}_i denote the i th col-

umn of \mathbf{C} and \mathbf{D} respectively. Hence, $\mathbf{c}_1, \dots, \mathbf{c}_n$ are linearly independent and so are $\mathbf{d}_1, \dots, \mathbf{d}_n$. Assume there exist $f_1(z), \dots, f_n(z) \in \mathbb{F}_{2^m}(z)$ such that $\sum_{i=1}^n f_i(z)(z\mathbf{c}_i - \mathbf{d}_i) = 0$. Without loss of generality, assume $f_i(z) = \frac{g_i(z)}{h(z)}$ for $i \in \{1, 2, \dots, n\}$, where $g_i(z), h(z) \in \mathbb{F}_{2^m}[z]$. Thus, $\sum_{i=1}^n g_i(z)(z\mathbf{c}_i - \mathbf{d}_i) = 0$. Let $k = \max_{i \in \{1, 2, \dots, n\}} \{d_{g_i}\}$ and assume $g_i(z) = \sum_{l=0}^k a_{l,i} z^l$. Then, it follows

$$\begin{aligned} \sum_{i=1}^n g_i(z)(z\mathbf{c}_i - \mathbf{d}_i) &= \sum_{l=0}^k \sum_{i=1}^n (a_{l,i} z^{l+1} \mathbf{c}_i - a_{l,i} z^l \mathbf{d}_i) \\ &= z^{k+1} \sum_{i=1}^n a_{k,i} \mathbf{c}_i + \sum_{l=0}^{k-1} z^{l+1} \sum_{i=1}^n (a_{l,i} \mathbf{c}_i - a_{l+1,i} \mathbf{d}_i) \\ &\quad - \sum_{i=1}^n a_{0,i} \mathbf{d}_i = 0 \end{aligned}$$

Therefore, the following equations must hold:

$$\begin{aligned} \sum_{i=1}^n a_{k,i} \mathbf{c}_i &= 0 \quad \sum_{i=1}^n a_{0,i} \mathbf{d}_i = 0 \\ \sum_{i=1}^n (a_{l,i} \mathbf{c}_i - a_{l+1,i} \mathbf{d}_i) &= 0 \quad \forall l \in \{0, \dots, k-1\} \end{aligned}$$

Thus $a_{l,i} = 0$ for any $i \in \{1, \dots, n\}, l \in \{0, \dots, k\}$, implying $f_i(z) = 0$. Hence, $\text{rank}(z\mathbf{C} - \mathbf{D}) = n$.

Then, there must be an $n \times n$ invertible submatrix in $z\mathbf{C} - \mathbf{D}$. Without loss of generality, assume this submatrix consists of the top n rows of $z\mathbf{C} - \mathbf{D}$ and denote this submatrix by \mathbf{E}_{n+1} . Let \mathbf{b} denote the $(n+1)$ th row of $z\mathbf{C} - \mathbf{D}$. In order to get a non-zero solution to equation (25), we first fix $r_{n+1}(z) = -1$. Therefore, equation (25) is transformed into $(r_1(z), \dots, r_n(z))\mathbf{E}_{n+1} = \mathbf{b}$. Let \mathbf{E}_i denote the submatrix acquired by replacing the i th row of \mathbf{E}_{n+1} with \mathbf{b} . Hence, we get a non-zero solution to (25), $\mathbf{r}(z) = (\frac{\det \mathbf{E}_1}{\det \mathbf{E}_{n+1}}, \dots, \frac{\det \mathbf{E}_n}{\det \mathbf{E}_{n+1}}, -1)$. Moreover, $\bar{\mathbf{r}}(z) = (\det \mathbf{E}_1, \dots, \det \mathbf{E}_n, -\det \mathbf{E}_{n+1})$ is also a solution. Note that the degree of z in each $\det \mathbf{E}_i$ is at most n . Thus, $\bar{\mathbf{r}}(z)$ can be formulated as $(1, z, \dots, z^n)\mathbf{F}$, where \mathbf{F} is an $(n+1) \times (n+1)$ matrix. Since $\text{rank}(z\mathbf{C} - \mathbf{D}) = n$, all the solutions to equation (25) form a one-dimensional linear space. Thus, all solutions must be linearly dependent on $\bar{\mathbf{r}}(z)$. ■

Based on Lemma B.2, we can easily derive that each \mathbf{V}_1 satisfying Eq. (5) is related to \mathbf{V}_1^* through a transform equation, as defined in Lemma VI.1.

Proof of Lemma VI.1: Let \mathbf{r}_i be the i th row of \mathbf{V}_1 , which satisfies Eq. (5). According to Lemma B.2, \mathbf{r}_i must have the form $f_i(\eta(\mathbf{x}^i))(1, \eta(\mathbf{x}^i), \dots, \eta^n(\mathbf{x}^i))\mathbf{F}$, where $f_i(z)$ is a non-zero rational function in $\mathbb{F}_{2^m}(z)$. Hence, \mathbf{V}_1 can be written as $\mathbf{G}\mathbf{V}_1^*\mathbf{F}$. Moreover, Eq. (25) can be rewritten as follows:

$$(z, z^2, \dots, z^{n+1})\mathbf{F}\mathbf{C} = (1, z, \dots, z^n)\mathbf{F}\mathbf{B}\mathbf{A}$$

The right side of the above equation contains no z^{n+1} , and thus the $(n+1)$ th row of $\mathbf{F}\mathbf{C}$ must be zero. Similarly, there is no constant term on the left side of the above equation, implying that the 1st row of $\mathbf{F}\mathbf{B}\mathbf{A}$ is zero. ■

Now assume the coupling relation $p_i(\mathbf{x}) = \frac{f(\eta(\mathbf{x}))}{g(\eta(\mathbf{x}))} \in \mathcal{S}_i'$ holds for the network. We will prove that \mathcal{B}_i' is violated

for $n > 1$, and thus it is impossible for the three unicast sessions to asymptotically achieve the rate tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ via PBNA. Without loss of generality, assume $f(z) = \sum_{k=0}^{n-1} a_k z^k$ and $g(z) = \sum_{k=0}^{n-1} b_k z^k$. Apparently, if $\text{rank}(\mathbf{V}_1) < n + 1$, \mathcal{B}'_i is violated. Thus, in the rest of this proof, we assume $\text{rank}(\mathbf{V}_1) = n + 1$. By Lemma VI.1, $\mathbf{V}_1 = \mathbf{G}\mathbf{V}_1^*\mathbf{F}$, where \mathbf{F} is an $(n + 1) \times (n + 1)$ invertible matrix. The j th row of \mathbf{V}_1 is $\mathbf{r}_j = f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^n(\mathbf{x}^j))\mathbf{F}$. Since the $(n + 1)$ th row of $\mathbf{F}\mathbf{C}$ is zero, we have $\mathbf{r}_j\mathbf{C} = f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^{n-1}(\mathbf{x}^j))\mathbf{H}$, where \mathbf{H} consists of the top n rows of $\mathbf{F}\mathbf{C}$ and $\text{rank}(\mathbf{H}) = n$. Let $\mathbf{a} = (a_0, a_1, \dots, a_n)^T$ and $\mathbf{b} = (b_0, b_1, \dots, b_{n-1})^T$. For $i = 1, 2$, we define $\mathbf{a}' = \mathbf{F}^{-1}\mathbf{a}$ and $\mathbf{b}' = \mathbf{H}^{-1}\mathbf{b}$. It follows

$$\begin{aligned} \mathbf{r}_j\mathbf{a}' &= f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^n(\mathbf{x}^j))\mathbf{F}\mathbf{a}' \\ &= f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^n(\mathbf{x}^j))\mathbf{a} \\ &= f_j(\eta(\mathbf{x}^j))f(\eta(\mathbf{x}^j)) \\ &= f_j(\eta(\mathbf{x}^j))p_i(\mathbf{x}^j)g(\eta(\mathbf{x}^j)) \\ &= p_i(\mathbf{x}^j)f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^{n-1}(\mathbf{x}^j))\mathbf{b} \\ &= p_i(\mathbf{x}^j)f_j(\eta(\mathbf{x}^j))(1, \eta(\mathbf{x}^j), \dots, \eta^{n-1}(\mathbf{x}^j))\mathbf{H}\mathbf{b}' \\ &= p_i(\mathbf{x}^j)\mathbf{r}_j\mathbf{C}\mathbf{b}' \end{aligned}$$

Hence, the columns of $(\mathbf{V}_1 \quad \mathbf{P}_i\mathbf{V}_1\mathbf{C})$ are linearly dependent, violating \mathcal{B}'_i . Similarly, we can prove the case of $i = 3$.

APPENDIX C

PROOFS OF INTERPRETATIONS OF FEASIBILITY CONDITIONS OF PBNA

A. $\eta(\mathbf{x}) = 1$

First, note that $\eta(\mathbf{x})$ can be rewritten as a ratio of two rational functions $\eta(\mathbf{x}) = \frac{f_{213}(\mathbf{x})}{f_{312}(\mathbf{x})}$, where $f_{ijk}(\mathbf{x}) \triangleq \frac{m_{ij}(\mathbf{x})m_{jk}(\mathbf{x})}{m_{ik}(\mathbf{x})}$. Hence, in order to interpret $\eta(\mathbf{x}) = 1$, we first study the properties of $f_{ijk}(\mathbf{x})$.

The following lemma is to be used to derive the general structure of $f_{ijk}(\mathbf{x})$. Basically, it provides an easy method to calculate the greatest common divisor of two transfer functions with one common starting edge or ending edge.

Lemma C.1. *The following statements hold:*

- 1) For $e_1, e_2, e_3 \in E'$ such that e_2, e_3 are both downstream of e_1 . Let e be the last edge of the topological ordering of the edges in $\mathcal{C}_{e_1e_2} \cap \mathcal{C}_{e_1e_3}$. Then $m_{e_1e}(\mathbf{x}) = \text{gcd}(m_{e_1e_2}(\mathbf{x}), m_{e_1e_3}(\mathbf{x}))$.
- 2) For $e_1, e_2, e_3 \in E'$ such that e_1, e_2 are both upstream of e_3 . Let e be the first edge of the topological ordering of the edges in $\mathcal{C}_{e_1e_3} \cap \mathcal{C}_{e_2e_3}$. Then $m_{ee_3}(\mathbf{x}) = \text{gcd}(m_{e_1e_3}(\mathbf{x}), m_{e_2e_3}(\mathbf{x}))$.

Proof: First, consider the first statement. By Lemma A.3, the following equations hold: $m_{e_1e_2}(\mathbf{x}) = m_{e_1e}(\mathbf{x})m_{ee_2}(\mathbf{x})$ and $m_{e_1e_3}(\mathbf{x}) = m_{e_1e}(\mathbf{x})m_{ee_3}(\mathbf{x})$. Thus $m_{e_1e}(\mathbf{x}) \mid \text{gcd}(m_{e_1e_2}(\mathbf{x}), m_{e_1e_3}(\mathbf{x}))$. Assume $\text{gcd}(m_{ee_2}(\mathbf{x}), m_{ee_3}(\mathbf{x})) \neq 1$. By Lemma A.3, there exists bottlenecks e_4, e_5 such that $m_{e_4e_5}(\mathbf{x}) \mid \text{gcd}(m_{ee_2}(\mathbf{x}), m_{ee_3}(\mathbf{x}))$. Clearly, $e_5 \in \mathcal{C}_{e_1e_2} \cap \mathcal{C}_{e_1e_3}$ and e_5 is downstream of e , which contradicts that e is the last edge of the topological ordering of $\mathcal{C}_{e_1e_2} \cap \mathcal{C}_{e_1e_3}$. Hence, we have proved that $\text{gcd}(m_{ee_2}(\mathbf{x}), m_{ee_3}(\mathbf{x})) = 1$, which in turn

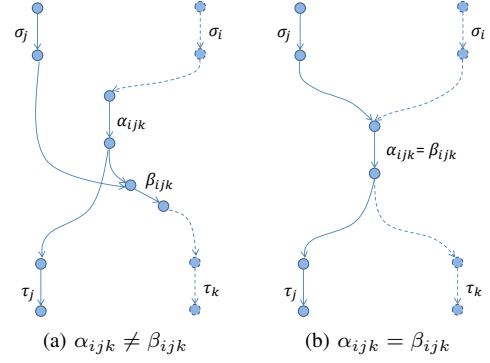


Fig. 9. The structure of $f_{ijk}(\mathbf{x})$ can be classified into two types: 1) $\alpha_{ijk} \neq \beta_{ijk}$ such that $f_{ijk}(\mathbf{x})$ is a rational function with non-constant denominator; 2) $\alpha_{ijk} = \beta_{ijk}$ such that $f_{ijk}(\mathbf{x})$ is a polynomial.

implies that $m_{e_1e}(\mathbf{x}) = \text{gcd}(m_{e_1e_2}(\mathbf{x}), m_{e_1e_3}(\mathbf{x}))$. Similarly, we can prove the other statement. ■

Using the above lemma, $f_{ijk}(\mathbf{x})$ can be reformulated as a fraction of two coprime polynomials, as shown below.

Corollary C.1. *$f_{ijk}(\mathbf{x})$ can be formulated as*

$$f_{ijk}(\mathbf{x}) = \frac{m_{\sigma_j, \beta_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x})}{m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})} \quad (31)$$

where $\text{gcd}(m_{\sigma_j, \beta_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x}), m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})) = 1$.

Proof: $f_{ijk}(\mathbf{x})$ can be calculated as

$$\begin{aligned} f_{ijk}(\mathbf{x}) &= \frac{m_{\sigma_i, \alpha_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x})m_{jk}(\mathbf{x})}{m_{\sigma_i, \alpha_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_k}(\mathbf{x})} \\ &= \frac{m_{\alpha_{ijk}, \tau_j}(\mathbf{x})m_{jk}(\mathbf{x})}{m_{\alpha_{ijk}, \tau_k}(\mathbf{x})} \\ &= \frac{m_{\alpha_{ijk}, \tau_j}(\mathbf{x})m_{\sigma_j, \beta_{ijk}}(\mathbf{x})m_{\beta_{ijk}, \tau_k}(\mathbf{x})}{m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})m_{\beta_{ijk}, \tau_k}(\mathbf{x})} \\ &= \frac{m_{\sigma_j, \beta_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x})}{m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})} \end{aligned}$$

By Lemma , $\text{gcd}(m_{\alpha_{ijk}, \tau_k}(\mathbf{x}), m_{\alpha_{ijk}, \tau_j}(\mathbf{x})) = 1$ and thus $\text{gcd}(m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x}), m_{\alpha_{ijk}, \tau_j}(\mathbf{x})) = 1$. Meanwhile, $\text{gcd}(m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x}), m_{\sigma_j, \beta_{ijk}}(\mathbf{x})) = 1$. Hence, we must have $\text{gcd}(m_{\sigma_j, \beta_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x}), m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})) = 1$. ■

According to Corollary C.1, the structure of $f_{ijk}(\mathbf{x})$ must fall into one of the two types, as shown in Fig. 9. In Fig. 9a, $\alpha_{ijk} \neq \beta_{ijk}$ and $f_{ijk}(\mathbf{x})$ is a rational function, the denominator of which is a non-constant polynomial $m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x})$. On the other hand, when $\alpha_{ijk} \in \mathcal{C}_{jk}$ and thus $\alpha_{ijk} = \beta_{ijk}$, as shown in Fig. 9b, $f_{ijk}(\mathbf{x})$ becomes a polynomial $m_{\sigma_j, \alpha_{ijk}}(\mathbf{x})m_{\alpha_{ijk}, \tau_j}(\mathbf{x})$.

Moreover, using Corollary C.1, we can easily check whether two $f_{ijk}(\mathbf{x})$'s are equivalent, as shown in the next corollary. It is easy to see that Theorem VII.1 is just a special case of this corollary.

Corollary C.2. *Assume $i, j, k, i', k' \in \{1, 2, 3\}$ such that $i \neq j, j \neq k$ and $i' \neq j, j \neq k'$. $f_{ijk}(\mathbf{x}) = f_{i'jk'}(\mathbf{x})$ if and only if $\alpha_{ijk} = \alpha_{i'jk'}$ and $\beta_{ijk} = \beta_{i'jk'}$.*

Proof: By Corollary C.1, if $\alpha_{ijk} = \alpha_{i'jk'}$ and $\beta_{ijk} =$

$\beta_{i'jk'}$, we must have $f_{ijk}(\mathbf{x}) = f_{i'jk'}(\mathbf{x})$. Conversely, if $f_{ijk}(\mathbf{x}) = f_{i'jk'}(\mathbf{x})$, $m_{\alpha_{ijk}, \beta_{ijk}}(\mathbf{x}) = m_{\alpha_{i'jk'}, \beta_{i'jk'}}(\mathbf{x})$. Thus $\alpha_{ijk} = \alpha_{i'jk'}$ and $\beta_{ijk} = \beta_{i'jk'}$ by Lemma A.2. ■

B. $p_i(\mathbf{x}) = 1$ and $p_i(\mathbf{x}) = \eta(\mathbf{x})$

Using Lemma A.1, we can easily prove Theorem VII.2, as shown below.

Proof of Theorem VII.2: Apparently, by Lemma A.1 and the definition of $p_1(\mathbf{x})$, $p_1(\mathbf{x}) = 1$ if and only if the minimum cut separating σ_1, σ_2 from τ_1 and τ_3 is one, i.e., $C_{12,13} = 1$. In order to interpret $p_1(\mathbf{x}) = \eta(\mathbf{x})$, we consider $q_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{p_1(\mathbf{x})} = \frac{m_{11}(\mathbf{x})m_{32}(\mathbf{x})}{m_{12}(\mathbf{x})m_{31}(\mathbf{x})}$. Hence $p_1(\mathbf{x}) = \eta(\mathbf{x})$ is equivalent to $q_1(\mathbf{x}) = 1$. Similarly, using Lemma A.1, it is easy to see that $p_1(\mathbf{x}) = \eta(\mathbf{x})$ if and only if the minimum cut separating σ_1, σ_3 from τ_1, τ_2 is one, i.e., $C_{13,12} = 1$. ■

C. $p_1(\mathbf{x}) = \frac{\eta(\mathbf{x})}{1+\eta(\mathbf{x})}$ and $p_2(\mathbf{x}), p_3(\mathbf{x}) = 1 + \eta(\mathbf{x})$

Note that the three coupling relations can be respectively reformulated in terms of $f_{ijk}(\mathbf{x})$ as follows:

$$\begin{aligned} m_{11}(\mathbf{x}) &= f_{312}(\mathbf{x}) + f_{213}(\mathbf{x}) \\ m_{22}(\mathbf{x}) &= f_{123}(\mathbf{x}) + f_{321}(\mathbf{x}) \\ m_{33}(\mathbf{x}) &= f_{231}(\mathbf{x}) + f_{132}(\mathbf{x}) \end{aligned}$$

Thus, as shown below, the three coupling relations can also be interpreted by using the properties of $f_{ijk}(\mathbf{x})$.

Proof of Theorem VII.3: We only prove statement 1). The other statements can be proved similarly. First, we prove the “if” part. Due to $\alpha_{312} \in \mathcal{C}_{12}$ and $\alpha_{213} \in \mathcal{C}_{13}$, $f_{312}(\mathbf{x}) = m_{\sigma_1, \alpha_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})$ and $f_{213}(\mathbf{x}) = m_{\sigma_1, \alpha_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})$. Hence, $f_{312}(\mathbf{x}) + f_{213}(\mathbf{x}) = m_{\sigma_1, \alpha_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x}) + m_{\sigma_1, \alpha_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})$. On the other hand, because $\alpha_{312} \parallel \alpha_{213}$ and $\{\alpha_{312}, \alpha_{213}\}$ forms a cut which separates σ_1 from τ_1 , $m_{11}(\mathbf{x}) = m_{\sigma_1, \alpha_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x}) + m_{\sigma_1, \alpha_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})$ by Lemma A.4. Therefore, $m_{11}(\mathbf{x}) = f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$.

Next we prove the “only if” part. Assume $m_{11}(\mathbf{x}) = f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$. If $\alpha_{312} \notin \mathcal{C}_{12}$ but $\alpha_{213} \in \mathcal{C}_{13}$, $f_{312}(\mathbf{x})$ is a rational function whose denominator is a non-constant polynomial, while $f_{213}(\mathbf{x})$ is a polynomial. Hence $f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$ must be a rational function with non-constant denominator, and thus $m_{11}(\mathbf{x}) \neq f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$. Similarly, if $\alpha_{312} \in \mathcal{C}_{12}$ but $\alpha_{213} \notin \mathcal{C}_{13}$, we can also prove that $m_{11}(\mathbf{x}) \neq f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$.

Now assume $\alpha_{312} \notin \mathcal{C}_{12}$ and $\alpha_{213} \notin \mathcal{C}_{13}$. It follows that $f_{312}(\mathbf{x}) = \frac{m_{\sigma_1, \beta_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})}{m_{\alpha_{312}, \beta_{312}}(\mathbf{x})}$ and $f_{213}(\mathbf{x}) = \frac{m_{\sigma_1, \beta_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})}{m_{\alpha_{213}, \beta_{213}}(\mathbf{x})}$. Because $\eta(\mathbf{x}) \neq 1$, we have $f_{312}(\mathbf{x}) \neq f_{213}(\mathbf{x})$, which indicates that $\alpha_{312} \neq \alpha_{213}$ or $\beta_{312} \neq \beta_{213}$ by Corollary C.2, and $m_{\alpha_{312}, \beta_{312}}(\mathbf{x}) \neq m_{\alpha_{213}, \beta_{213}}(\mathbf{x})$. Therefore, by Lemma A.3, one of the following cases must hold: 1) There exists an irreducible polynomial $m_{ee'}(\mathbf{x})$ such that $m_{ee'}(\mathbf{x}) \mid m_{\alpha_{312}, \beta_{312}}(\mathbf{x})$ but $m_{ee'}(\mathbf{x}) \nmid m_{\alpha_{213}, \beta_{213}}(\mathbf{x})$; 2) there exists an irreducible polynomial $m_{ee'}(\mathbf{x})$ such that $m_{ee'}(\mathbf{x}) \nmid m_{\alpha_{312}, \beta_{312}}(\mathbf{x})$ but $m_{ee'}(\mathbf{x}) \mid m_{\alpha_{213}, \beta_{213}}(\mathbf{x})$.

Consider case 1). Define the following polynomials: $f(\mathbf{x}) = \text{lcm}(m_{\alpha_{312}, \beta_{312}}(\mathbf{x}), m_{\alpha_{213}, \beta_{213}}(\mathbf{x}))$ ⁵ and $f_1(\mathbf{x}) = f(\mathbf{x})/m_{\alpha_{312}, \beta_{312}}(\mathbf{x})$ and $f_2(\mathbf{x}) = f(\mathbf{x})/m_{\alpha_{213}, \beta_{213}}(\mathbf{x})$. Hence, we have $m_{ee'}(\mathbf{x}) \nmid f_1(\mathbf{x})$, $m_{ee'}(\mathbf{x}) \mid f_2(\mathbf{x})$, and $f_{312}(\mathbf{x}) + f_{213}(\mathbf{x}) = [m_{\sigma_1, \beta_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})f_1(\mathbf{x}) + m_{\sigma_1, \beta_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})f_2(\mathbf{x})]/f(\mathbf{x})$. Moreover, due to $\gcd(m_{\alpha_{312}, \beta_{312}}(\mathbf{x}), m_{\sigma_1, \beta_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})) = 1$, it follows that $m_{ee'}(\mathbf{x}) \nmid m_{\sigma_1, \beta_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})$. This implies that $m_{ee'}(\mathbf{x}) \nmid m_{\sigma_1, \beta_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x})f_1(\mathbf{x}) + m_{\sigma_1, \beta_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})f_2(\mathbf{x})$. However, $m_{ee'}(\mathbf{x}) \mid f(\mathbf{x})$. This indicates that $f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$ is a rational function with non-constant denominator. Thus $m_{11}(\mathbf{x}) \neq f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$. Similarly, for case 2), we can also prove that $m_{11}(\mathbf{x}) \neq f_{312}(\mathbf{x}) + f_{213}(\mathbf{x})$.

Thus, we have proved that $\alpha_{312} \in \mathcal{C}_{12}$ and $\alpha_{213} \in \mathcal{C}_{13}$. It immediately follows that $m_{11}(\mathbf{x}) = m_{\sigma_1, \alpha_{312}}(\mathbf{x})m_{\alpha_{312}, \tau_1}(\mathbf{x}) + m_{\sigma_1, \alpha_{213}}(\mathbf{x})m_{\alpha_{213}, \tau_1}(\mathbf{x})$. Hence each path P in $\mathcal{P}_{\sigma_1 \tau_1}$ either pass through α_{312} or α_{213} , implying that $\{\alpha_{312}, \alpha_{213}\}$ forms a cut separating σ_1 from τ_1 . Moreover, according to Lemma A.4, $\alpha_{312} \parallel \alpha_{213}$. ■

APPENDIX D

PROOFS OF LEMMAS ON MULTIVARIATE POLYNOMIALS

In this section, we present the proof of Lemma B.1. We first prove that Lemma B.1 holds for the case where $s(\mathbf{x})$ and $t(\mathbf{x})$ are both univariate polynomials. In order to extend this result to multivariate polynomials, we employ a simple idea that each multivariate polynomial can be viewed as an equivalent univariate polynomial on a field of rational functions. Specifically, we prove that the problem of checking if two multivariate polynomials are co-prime is equivalent to checking if their equivalent univariate polynomials are co-prime. Finally, based on this result, we prove that Lemma B.1 also holds for the multivariate case.

A. The Univariate Case

In the following lemma, we show that Lemma B.1 holds for the univariate case.

Lemma D.1. *Let \mathbb{F} be a field, and z, y are two variables. Consider four non-zero polynomials $f(z), g(z) \in \mathbb{F}[z]$ and $s(y), t(y) \in \mathbb{F}[y]$, such that $\gcd(f(z), g(z)) = 1$ and $\gcd(s(y), t(y)) = 1$. Denote $d = \max\{d_f, d_g\}$. Define two polynomials $\alpha(y) = f(\frac{s(y)}{t(y)})t^d(y)$ and $\beta(y) = g(\frac{s(y)}{t(y)})t^d(y)$. Then $\gcd(\alpha(y), \beta(y)) = 1$.*

Proof: Assume $w(x) = \gcd(\alpha(x), \beta(x))$ is non-trivial. Thus we can find an extension field \mathbb{F} of \mathbb{F} such that there exists $x_0 \in \mathbb{F}$ which satisfies $w(x_0) = 0$ and hence $\alpha(x_0) = \beta(x_0) = 0$. In the rest of this proof, we restrict our discussion in \mathbb{F} . Note that $\gcd(f(z), g(z)) = 1$ and $\gcd(s(x), t(x)) = 1$ also hold for \mathbb{F} . Assume $t(x_0) = 0$ and thus $x - x_0 \mid t(x)$. Since $\gcd(s(x), t(x)) = 1$, it follows that $x - x_0 \nmid s(x)$ and thus $s(x_0) \neq 0$. Hence, either $\alpha(x_0) \neq 0$ or $\beta(x_0) \neq 0$, contradicting that $\alpha(x_0), \beta(x_0)$

⁵We use $\text{lcm}(f(\mathbf{x}), g(\mathbf{x}))$ to denote the least common multiple of two polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$.

are both zeros. Hence, we have proved that $t(x_0) \neq 0$. Then we have $f(\frac{s(x_0)}{t(x_0)}) = \frac{\alpha(x_0)}{t^d(x_0)} = 0$ and $g(\frac{s(x_0)}{t(x_0)}) = \frac{\beta(x_0)}{t^d(x_0)} = 0$, which implies that $z - \frac{s(x_0)}{t(x_0)}$ is a common divisor of $f(z)$ and $g(z)$, contradicting $\gcd(f(z), g(z)) = 1$. Thus, we have proved that $\gcd(\alpha(y), \beta(y)) = 1$. ■

B. Viewing Multivariate as Univariate

In order to extend Lemma D.1 to the multivariate case, we first show that each multivariate polynomial can be viewed as an equivalent univariate polynomial on a field of rational functions. Let $\mathbf{y} = (y_1, y_2, \dots, y_k)$ be a vector of variables. For any $i \in \{1, 2, \dots, k\}$, define $\mathbf{y}_i = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k)$, i.e., the vector consisting of all variables in \mathbf{y} other than y_i . Note that any polynomial $f(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ can be formulated as $f(\mathbf{y}) = f_0(\mathbf{y}_i) + f_1(\mathbf{y}_i)y_i + \dots + f_p(\mathbf{y}_i)y_i^p$, where each $f_j(\mathbf{y}_i)$ is a polynomial in $\mathbb{F}[\mathbf{y}_i]$. Because $\mathbb{F}[\mathbf{y}_i]$ is a subset of $\mathbb{F}(\mathbf{y}_i)$, $f(\mathbf{y})$ can also be viewed as a univariate polynomial in $\mathbb{F}(\mathbf{y}_i)[y_i]$. We use $f(y_i)$ to denote $f(\mathbf{y})$'s equivalent counterpart in $\mathbb{F}(\mathbf{y}_i)[y_i]$. To differentiate these two concepts, we reserve the notations, such as “|”, “gcd” and “lcm” for field \mathbb{F} , and append “1” as a subscript to these notations to suggest they are specific to field $\mathbb{F}(\mathbf{y}_i)$. For example, for $f(\mathbf{y}), g(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ and $u(y_i), v(y_i) \in \mathbb{F}(\mathbf{y}_i)[y_i]$, $g(\mathbf{y}) \mid f(\mathbf{y})$ means that there exists $h(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ such that $f(\mathbf{y}) = h(\mathbf{y})g(\mathbf{y})$, and $u(y_i) \mid_1 v(y_i)$ means that there exists $w(y_i) \in \mathbb{F}[\mathbf{y}_i][y_i]$ such that $v(y_i) = w(y_i)u(y_i)$.

Lemma D.2. Assume $g(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$ and $f(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ is of the form $f(\mathbf{y}) = \sum_{j=0}^p f_j(\mathbf{y}_i)y_i^j$, where $f_j(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$. Then $g(\mathbf{y}_i) \mid f(\mathbf{y})$ if and only if $g(\mathbf{y}_i) \mid f_j(\mathbf{y}_i)$ for each $j \in \{0, 1, \dots, p\}$.

Proof: Apparently, if $g(\mathbf{y}_i) \mid f_j(\mathbf{y}_i)$ for any $j \in \{0, 1, \dots, p\}$, $g(\mathbf{y}_i) \mid f(\mathbf{y})$. Now assume $g(\mathbf{y}_i) \mid f(\mathbf{y})$. Thus there exists $h(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ such that $f(\mathbf{y}) = g(\mathbf{y}_i)h(\mathbf{y})$. Let $h(\mathbf{y}) = \sum_{j=0}^p h_j(\mathbf{y}_i)y_i^j$. Hence, it follows that $f_j(\mathbf{y}_i) = h_j(\mathbf{y}_i)g(\mathbf{y}_i)$ and thus $g(\mathbf{y}_i) \mid f_j(\mathbf{y}_i)$. ■

The following result follows immediately from Lemma D.2.

Corollary D.1. Let $g(\mathbf{y}_i)$ and $f(\mathbf{y})$ be defined as Lemma D.2. Then $\gcd(g(\mathbf{y}_i), f(\mathbf{y})) = \gcd(g(\mathbf{y}_i), f_0(\mathbf{y}_i), \dots, f_p(\mathbf{y}_i))$.

Proof: Note that any divisor of $g(\mathbf{y}_i)$ must be a polynomial in $\mathbb{F}[\mathbf{y}_i]$. Let $d(\mathbf{y}_i) = \gcd(g(\mathbf{y}_i), f(\mathbf{y}))$ and $d'(\mathbf{y}_i) = \gcd(g(\mathbf{y}_i), f_0(\mathbf{y}_i), \dots, f_p(\mathbf{y}_i))$. By Lemma D.2, $d(\mathbf{y}_i) \mid f_j(\mathbf{y}_i)$ for any $j \in \{0, 1, \dots, p\}$, implying that $d(\mathbf{y}_i) \mid d'(\mathbf{y}_i)$. On the other hand, $d'(\mathbf{y}_i) \mid f(\mathbf{y})$, and thus $d'(\mathbf{y}_i) \mid d(\mathbf{y}_i)$. Hence, $d(\mathbf{y}_i) = d'(\mathbf{y}_i)$. ■

Corollary D.2. For $t \in \{1, 2, \dots, s\}$, let $f_t(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ be defined as $f_t(\mathbf{y}) = \sum_{j=0}^{p_t} f_{tj}(\mathbf{y}_i)y_i^j$, where $f_{tj}(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$. Let $g(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$. It follows

$$\begin{aligned} & \gcd(g(\mathbf{y}_i), f_1(\mathbf{y}), \dots, f_t(\mathbf{y})) \\ &= \gcd(g(\mathbf{y}_i), f_{10}(\mathbf{y}_i), \dots, f_{1p_1}(\mathbf{y}_i), \dots, \\ & \quad f_{s0}(\mathbf{y}_i), \dots, f_{sp_s}(\mathbf{y}_i)) \end{aligned}$$

Proof: We have the following equations

$$\begin{aligned} & \gcd(g(\mathbf{y}_i), f_1(\mathbf{y}), \dots, f_t(\mathbf{y})) \\ &= \gcd(g(\mathbf{y}_i), f_1(\mathbf{y}), \dots, g(\mathbf{y}_i), f_t(\mathbf{y})) \\ &= \gcd(\gcd(g(\mathbf{y}_i), f_1(\mathbf{y})), \dots, \gcd(g(\mathbf{y}_i), f_s(\mathbf{y}))) \\ &= \gcd(g(\mathbf{y}_i), f_{10}(\mathbf{y}_i), \dots, f_{1p_1}(\mathbf{y}_i), \dots, \\ & \quad g(\mathbf{y}_i), f_{s0}(\mathbf{y}_i), \dots, f_{sp_s}(\mathbf{y}_i)) \\ &= \gcd(g(\mathbf{y}_i), f_{10}(\mathbf{y}_i), \dots, f_{1p_1}(\mathbf{y}_i), \dots, \\ & \quad f_{s0}(\mathbf{y}_i), \dots, f_{sp_s}(\mathbf{y}_i)) \end{aligned}$$

Lemma D.3. For $t \in \{1, 2, \dots, s\}$, let $a_t(\mathbf{y}), b_t(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$ such that $b_t(\mathbf{y}) \neq 0$ and $\gcd(a_t(\mathbf{y}), b_t(\mathbf{y})) = 1$. For $t \in \{1, 2, \dots, s\}$, let $v_t(\mathbf{y}) = \text{lcm}(b_1(\mathbf{y}), \dots, b_t(\mathbf{y}))$. Then we have

$$\gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_s(\mathbf{y})\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}, v_s(\mathbf{y})\right) = 1$$

Proof: We use induction on s to prove this lemma. Apparently, the lemma holds for $s = 1$ due to $\gcd(a_1(\mathbf{y}), b_1(\mathbf{y})) = 1$. Assume it holds for $s - 1$. Thus it follows

$$\begin{aligned} & \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_s(\mathbf{y})\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}, v_s(\mathbf{y})\right) \\ &= \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_s(\mathbf{y})\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}, b_s(\mathbf{y})\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &= \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, \gcd(a_s(\mathbf{y}), b_s(\mathbf{y}))\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &\stackrel{(a)}{=} \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_{s-1}(\mathbf{y})\frac{v_s(\mathbf{y})}{b_{s-1}(\mathbf{y})}, \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &\stackrel{(b)}{=} \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_{s-1}(\mathbf{y})\frac{v_s(\mathbf{y})}{b_{s-1}(\mathbf{y})}, \right. \\ & \quad \left. \gcd\left(v_{s-1}(\mathbf{y}), \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right)\right) \\ &= \gcd\left(a_1(\mathbf{y})\frac{v_s(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_{s-1}(\mathbf{y})\frac{v_s(\mathbf{y})}{b_{s-1}(\mathbf{y})}, v_{s-1}(\mathbf{y}), \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &= \gcd\left(\frac{v_s(\mathbf{y})}{v_{s-1}(\mathbf{y})} \gcd\left(a_1(\mathbf{y})\frac{v_{s-1}(\mathbf{y})}{b_1(\mathbf{y})}, \dots, a_{s-1}(\mathbf{y})\frac{v_{s-1}(\mathbf{y})}{b_{s-1}(\mathbf{y})}\right), \right. \\ & \quad \left. v_{s-1}(\mathbf{y}), \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &\stackrel{(c)}{=} \gcd\left(\frac{v_s(\mathbf{y})}{v_{s-1}(\mathbf{y})}, v_{s-1}(\mathbf{y}), \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})}\right) \\ &\stackrel{(d)}{=} \gcd\left(\frac{b_s(\mathbf{y})}{\gcd(v_{s-1}(\mathbf{y}), b_s(\mathbf{y}))}, v_{s-1}(\mathbf{y}), \frac{v_{s-1}(\mathbf{y})}{\gcd(v_{s-1}(\mathbf{y}), b_s(\mathbf{y}))}\right) \\ &= \gcd(1, v_{s-1}(\mathbf{y})) = 1 \end{aligned}$$

In the above equations, (a) is due to $\gcd(a_s(\mathbf{y}), b_s(\mathbf{y})) = 1$; (b) follows from the fact that $\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})} \mid v_{s-1}(\mathbf{y})$ and thus $\frac{v_s(\mathbf{y})}{b_s(\mathbf{y})} = \gcd(v_{s-1}(\mathbf{y}), \frac{v_s(\mathbf{y})}{b_s(\mathbf{y})})$; (c) follows from the inductive assumption; (d) is due to the equality: $v_s(\mathbf{y}) = \text{lcm}(v_{s-1}(\mathbf{y}), b_s(\mathbf{y})) = \frac{v_{s-1}(\mathbf{y})b_s(\mathbf{y})}{\gcd(v_{s-1}(\mathbf{y}), b_s(\mathbf{y}))}$. ■

In general, each polynomial $h(y_i) \in \mathbb{F}(\mathbf{y}_i)[y_i]$ is of the form $h(y_i) = \frac{a_0(\mathbf{y}_i)}{b_0(\mathbf{y}_i)} + \frac{a_1(\mathbf{y}_i)}{b_1(\mathbf{y}_i)}y_i + \dots + \frac{a_p(\mathbf{y}_i)}{b_p(\mathbf{y}_i)}y_i^p$, where for each $j \in \{0, 1, \dots, p\}$, $a_j(\mathbf{y}_i), b_j(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$, $b_j(\mathbf{y}_i) \neq 0$,

$\gcd(a_j(\mathbf{y}_i), b_j(\mathbf{y}_i)) = 1$, and $a_p(\mathbf{y}_i) \neq 0$. Note that for each y_i^j which is absent in $h(\mathbf{y}_i)$, we let $a_j(\mathbf{y}_i) = 0$ and $b_j(\mathbf{y}_i) = 1$. Moreover, define the following polynomial $\mu_h(\mathbf{y}_i) = \text{lcm}(b_0(\mathbf{y}_i), b_1(\mathbf{y}_i), \dots, b_p(\mathbf{y}_i))$.

Corollary D.3. For $j \in \{1, 2, \dots, s\}$, let $f_j(y_i) \in \mathbb{F}(\mathbf{y}_i)[y_i]$. Define $v(\mathbf{y}_i) = \text{lcm}(\mu_{f_1}(\mathbf{y}_i), \dots, \mu_{f_s}(\mathbf{y}_i))$ and $\bar{f}_j(\mathbf{y}) = v(\mathbf{y}_i)f_j(y_i)$. Thus $\gcd(v(\mathbf{y}_i), \bar{f}_1(\mathbf{y}), \dots, \bar{f}_s(\mathbf{y})) = 1$

Proof: Assume $f_j(y_i)$ has the following form:

$$f_j(y_i) = \frac{a_{j0}(\mathbf{y}_i)}{b_{j0}(\mathbf{y}_i)} + \frac{a_{j1}(\mathbf{y}_i)}{b_{j1}(\mathbf{y}_i)}y_i + \dots + \frac{a_{jp_j}(\mathbf{y}_i)}{b_{jp_j}(\mathbf{y}_i)}y_i^{p_j}$$

where for any $j \in \{1, 2, \dots, s\}$ and $t \in \{0, 1, \dots, p_j\}$, $a_{jt}(\mathbf{y}_i), b_{jt}(\mathbf{y}_i) \in \mathbb{F}[\mathbf{y}_i]$, $b_{jt}(\mathbf{y}_i) \neq 0$ and $\gcd(a_{jt}(\mathbf{y}_i), b_{jt}(\mathbf{y}_i)) = 1$. Apparently, $v(\mathbf{y}_i)$ is the least common multiple of all $b_{jt}(\mathbf{y}_i)$'s. Define $u_{jt}(\mathbf{y}_i) = \frac{v(\mathbf{y}_i)}{b_{jt}(\mathbf{y}_i)} \in \mathbb{F}[\mathbf{y}_i]$. Hence, we have $\bar{f}_j(\mathbf{y}) = \sum_{t=0}^{p_j} a_{jt}(\mathbf{y}_i)u_{jt}(\mathbf{y}_i)y_i^t$. Then it follows

$$\begin{aligned} & \gcd(v(\mathbf{y}_i), \bar{f}_1(\mathbf{y}), \dots, \bar{f}_s(\mathbf{y})) \\ & \stackrel{(a)}{=} \gcd(v(\mathbf{y}_i), a_{10}(\mathbf{y}_i)u_{10}(\mathbf{y}_i), \dots, a_{1p_1}(\mathbf{y}_i)u_{1p_1}(\mathbf{y}_i), \dots, \\ & \quad a_{s0}(\mathbf{y}_i)u_{s0}(\mathbf{y}_i), \dots, a_{sp_s}(\mathbf{y}_i)u_{sp_s}(\mathbf{y}_i)) \\ & \stackrel{(b)}{=} 1 \end{aligned}$$

where (a) is due to Corollary D.2 and (b) follows from Lemma D.3. ■

Generally, the definitions of division in $\mathbb{F}[\mathbf{y}]$ and $\mathbb{F}(\mathbf{y}_i)[y_i]$ are different. However, the following theorem reveals the two definitions are closely related.

Theorem D.1. Consider two polynomials $f(\mathbf{y}), g(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$, where $g(\mathbf{y}) \neq 0$. Then $g(\mathbf{y}) \mid f(\mathbf{y})$ if and only if $g(y_i) \mid_1 f(y_i)$ for every $i \in \{1, 2, \dots, k\}$.

Proof: The division equation between $f(y_i)$ and $g(y_i)$ is as follows

$$f(y_i) = h_i(y_i)g(y_i) + r_i(y_i) \quad (32)$$

where $h_i(y_i), r_i(y_i) \in \mathbb{F}(\mathbf{y}_i)[y_i]$, and either $r_i(y_i) = 0$ or $\deg r_i < \deg g$. Due to the uniqueness of Equation (32), $f(\mathbf{y}) \mid g(\mathbf{y})$ immediately implies that for any $i \in \{1, 2, \dots, k\}$, $r_i(y_i) = 0$ and thus $g(y_i) \mid_1 f(y_i)$.

Conversely, assume for every $i \in \{1, \dots, k\}$, $g(y_i) \mid_1 f(y_i)$ and hence $r_i(y_i) = 0$. Denote $\bar{h}_i(\mathbf{y}) = \mu_{h_i}(\mathbf{y}_i)h_i(y_i)$. Clearly, $\bar{h}_i(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$. Then, the following equation holds

$$\mu_{h_i}(\mathbf{y}_i)f(\mathbf{y}) = \bar{h}_i(\mathbf{y})g(\mathbf{y})$$

By Corollary D.3, $\gcd(\mu_{h_i}(\mathbf{y}_i), \bar{h}_i(\mathbf{y})) = 1$. Thus, $\mu_{h_i}(\mathbf{y}_i) \mid g(\mathbf{y})$. Define $\bar{g}(\mathbf{y}) = \frac{g(\mathbf{y})}{\mu_{h_i}(\mathbf{y}_i)}$. By Lemma D.2, $\bar{g}(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$. Define $u(\mathbf{y}) = \frac{g(\mathbf{y})}{\gcd(f(\mathbf{y}), g(\mathbf{y}))} \in \mathbb{F}[\mathbf{y}]$. It follows that

$$\begin{aligned} u(\mathbf{y}) &= \frac{g(\mathbf{y})}{\gcd(f(\mathbf{y}), g(\mathbf{y}))} \\ &= \frac{\mu_{h_i}(\mathbf{y}_i)\bar{g}(\mathbf{y})}{\gcd(\bar{h}_i(\mathbf{y})\bar{g}(\mathbf{y}), \mu_{h_i}(\mathbf{y}_i)\bar{g}(\mathbf{y}))} \\ &= \frac{\mu_{h_i}(\mathbf{y}_i)\bar{g}(\mathbf{y})}{\bar{g}(\mathbf{y})\gcd(\bar{h}_i(\mathbf{y}), \mu_{h_i}(\mathbf{y}_i))} \end{aligned}$$

$$\begin{aligned} &= \frac{\mu_{h_i}(\mathbf{y}_i)\bar{g}(\mathbf{y})}{\bar{g}(\mathbf{y})} \\ &= \mu_{h_i}(\mathbf{y}_i) \end{aligned}$$

Note that variable y_i is absent in $u(\mathbf{y})$. Because y_i can be any arbitrary variable in \mathbf{y} , it immediately follows that all the variables in \mathbf{y} must be absent in $u(\mathbf{y})$, implying that $u(\mathbf{y})$ is a constant in \mathbb{F} . Hence $g(\mathbf{y}) \mid f(\mathbf{y})$. ■

Moreover, in the next theorem, we will prove that checking if two multivariate polynomials are co-prime is equivalent to checking if their equivalent univariate polynomials are co-prime.

Theorem D.2. Let $f(\mathbf{y}), g(\mathbf{y})$ be two non-zero polynomials in $\mathbb{F}[\mathbf{y}]$. Then $\gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$ if and only if $\gcd_1(f(y_i), g(y_i)) = 1$ for any $i \in \{1, 2, \dots, k\}$.

Proof: First, assume for any $i \in \{1, 2, \dots, k\}$, $\gcd_1(f(y_i), g(y_i)) = 1$. We use contradiction to prove that $\gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$. Assume $u(\mathbf{y}) = \gcd(f(\mathbf{y}), g(\mathbf{y}))$ is not constant. Let y_i be a variable which is present in $u(\mathbf{y})$. By Theorem D.1, $u(y_i) \mid_1 f(y_i)$ and $u(y_i) \mid_1 g(y_i)$, which contradicts that $\gcd_1(f(y_i), g(y_i)) = 1$.

Then, assume $\gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$. We also use contradiction to prove that for any $i \in \{1, 2, \dots, k\}$, $\gcd_1(f(y_i), g(y_i)) = 1$. Assume there exists $i \in \{1, \dots, k\}$ such that $v(y_i) = \gcd_1(f(y_i), g(y_i))$ is non-trivial. Define $w(\mathbf{y}) = \mu_v(\mathbf{y}_i)v(y_i) \in \mathbb{F}[\mathbf{y}]$. Clearly, $w(y_i) \mid_1 f(y_i)$ and $w(y_i) \mid_1 g(y_i)$. Thus, there exists $p(y_i), q(y_i) \in \mathbb{F}(\mathbf{y}_i)[y_i]$ such that

$$f(y_i) = w(y_i)p(y_i) \quad g(y_i) = w(y_i)q(y_i)$$

Let $s(\mathbf{y}_i) = \text{lcm}(\mu_p(\mathbf{y}_i), \mu_q(\mathbf{y}_i))$. Define $\bar{p}(\mathbf{y}) = s(\mathbf{y}_i)p(y_i)$ and $\bar{q}(\mathbf{y}) = s(\mathbf{y}_i)q(y_i)$. Apparently, $\bar{p}(\mathbf{y}), \bar{q}(\mathbf{y}) \in \mathbb{F}[\mathbf{y}]$. It follows that

$$s(\mathbf{y}_i)f(\mathbf{y}) = w(\mathbf{y})\bar{p}(\mathbf{y}) \quad s(\mathbf{y}_i)g(\mathbf{y}) = w(\mathbf{y})\bar{q}(\mathbf{y})$$

Then the following equation holds

$$s(\mathbf{y}_i)\gcd(f(\mathbf{y}), g(\mathbf{y})) = w(\mathbf{y})\gcd(\bar{p}(\mathbf{y}), \bar{q}(\mathbf{y}))$$

Due to Corollary D.3, $\gcd(s(\mathbf{y}_i), \gcd(\bar{p}(\mathbf{y}), \bar{q}(\mathbf{y}))) = \gcd(s(\mathbf{y}_i), \bar{p}(\mathbf{y}), \bar{q}(\mathbf{y})) = 1$. Hence $s(\mathbf{y}_i) \mid w(\mathbf{y})$. Let $\bar{w}(\mathbf{y}) = \frac{w(\mathbf{y})}{s(\mathbf{y}_i)}$. According to Lemma D.2, $\bar{w}(\mathbf{y})$ is a non-trivial polynomial in $\mathbb{F}[\mathbf{y}]$. Thus, $\bar{w}(\mathbf{y}) \mid \gcd(f(\mathbf{y}), g(\mathbf{y}))$, contradicting $\gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$. ■

C. The Multivariate Case

Now, we are in the place of extending Lemma D.1 to the multivariate case.

Proof of Lemma B.1: Note that if we substitute \mathbb{F} with $\mathbb{F}(\mathbf{y}_i)$ and \gcd with \gcd_1 in Lemma D.1, the lemma also holds. Apparently, $f(z), g(z) \in \mathbb{F}(\mathbf{y}_i)[z]$. We will prove that $\gcd_1(f(z), g(z)) = 1$. By contradiction, assume $r(z) = \gcd_1(f(z), g(z)) \in \mathbb{F}(\mathbf{y}_i)[z]$ is non-trivial. Let $\bar{f}(z) = \frac{f(z)}{r(z)}$ and $\bar{g}(z) = \frac{g(z)}{r(z)}$. Clearly, $\bar{f}(z)$ and $\bar{g}(z)$ are both non-zero polynomials in $\mathbb{F}(\mathbf{y}_i)[z]$. Then we can find an assignment to \mathbf{y}_i , denoted by \mathbf{y}_i^* , such that the coefficients of the maximum powers of z in $r(z), \bar{f}(z)$ and $\bar{g}(z)$ are all non-zeros. Let $\bar{r}(z)$

denote the univariate polynomial acquired by assigning $y_i = y_i^*$ to $r(z)$. Clearly, $\bar{r}(z)$ is a common divisor of $f(z)$ and $g(z)$ in $\mathbb{F}[z]$, contradicting $\gcd(f(z), g(z)) = 1$. Moreover, due to $\gcd(s(y), t(y)) = 1$ and Theorem D.2, $\gcd_1(s(y_i), t(y_i)) = 1$. Thus, by Lemma D.1, $\gcd_1(\alpha(y_i), \beta(y_i)) = 1$. Since i can be any integer in $\{1, 2, \dots, k\}$, it follows that $\gcd(\alpha(y), \beta(y)) = 1$ by Theorem D.2. ■

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